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Rough sets: Some extensions

Zdzisław Pawlak, Andrzej Skowron *

Institute of Mathematics, Warsaw University, Banacha 2, 02-097 Warsaw, Poland

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Commemorating the life and work of Zdzisław Pawlak.

Abstract

In this article, we present some extensions of the rough set approach and we outline a challenge for the rough set based research.

Keywords: Vague concepts; Indiscernibility; Discernibility; Generalized approximation spaces; Neighborhoods; Rough inclusion; Rough sets; Classifiers; Information granulation; Vague concept approximations; Vague dependency approximations; Rough mereology; Ontology approximation

The central problem of our age is how to act decisively in the absence of certainty.
Bertrand Russell (1950). An Inquiry into Meaning and Truth.
George Allen and Unwin, London; W.W. Norton, New York.

1. Introduction

We use notation introduced in [47]. The reader is also referred to the literature cited in [47].

The basic notions of rough sets and approximation spaces were introduced during the early 1980s (see, e.g., [42–44]). In this section, we give some introductory remarks on rough sets.

Rough set theory, proposed by Pawlak in 1982 [44,45], can be seen as a new mathematical approach to vagueness. The rough set philosophy is founded on the assumption that with every object of the universe of discourse we associate some information (data, knowledge). For example, if objects are patients suffering from a certain disease, symptoms of the disease form information about patients. Objects characterized by the same information are indiscernible (similar) in view of the available information about them. The

* Corresponding author.
E-mail address: skowron@mimuw.edu.pl (A. Skowron).

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indiscernibility relation generated in this way is the mathematical basis of rough set theory. This understanding of indiscernibility is related to the idea of Gottfried Wilhelm Leibniz that objects are indiscernible if and only if all available functionals take on identical values (Leibniz's Law of Indiscernibility: The Identity of Indiscernibles) [2,28]. However, in the rough set approach, indiscernibility is defined relative to a given set of functionals (attributes).

Any set of all indiscernible (similar) objects is called an elementary set, and forms a basic granule (atom) of knowledge about the universe. Any union of some elementary sets is referred to as a crisp (precise) set. A set which is not crisp is called rough (imprecise, vague).

Consequently, each rough set has boundary region cases, i.e., objects which cannot with certainty be classified either as members of the set or of its complement. Obviously crisp sets have no boundary region elements at all. This means that boundary region cases cannot be properly classified by employing available knowledge.

Thus, the assumption that objects can be "seen" only through the information available about them leads to the view that knowledge has a granular structure. Due to the granularity of knowledge, some objects of interest cannot be discerned and appear as the same (or similar). As a consequence, vague concepts, in contrast to precise concepts, cannot be characterized in terms of information about their elements. Therefore, in the proposed approach, we assume that any vague concept is replaced by a pair of precise concepts – called the lower and the upper approximation of the vague concept. The lower approximation consists of all objects which surely belong to the concept and the upper approximation contains all objects which possibly belong to the concept. The difference between the upper and the lower approximation constitutes the boundary region of the vague concept. These approximations are two basic operations in rough set theory.

Hence, rough set theory expresses vagueness not by means of membership, but by employing a boundary region of a set. If the boundary region of a set is empty, it means that the set is crisp, otherwise the set is rough (inexact). A nonempty boundary region of a set means that our knowledge about the set is not sufficient to define the set precisely.

Rough set theory is not an alternative to but rather is embedded in classical set theory. Rough set theory can be viewed as a specific implementation of Frege's idea of vagueness, i.e., imprecision in this approach is expressed by a boundary region of a set.

Rough set theory has attracted worldwide attention of many researchers and practitioners, who have contributed essentially to its development and applications. Rough set theory overlaps with many other theories. Despite this, rough set theory may be considered as an independent discipline in its own right. The rough set approach seems to be of fundamental importance in artificial intelligence and cognitive sciences, especially in research areas such as machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, and expert systems. The main advantage of rough set theory in data analysis is that it does not need any preliminary or additional information about data like probability distributions in statistics, basic probability assignments in Dempster–Shafer theory, a grade of membership or the value of possibility in fuzzy set theory (see, e.g., [13] where some combinations of rough sets with non-parametric statistics are studied).

This paper is structured as follows. In Section 2 we discuss some extensions of the basic rough set approach. First, in Section 2.1 we present approximation spaces based on neighborhoods of objects and inclusion measures of neighborhoods into concepts. Illustrative examples of such approximation spaces are included. In Section 2.2 inductive extensions of approximation spaces are defined. Such extensions are used in approximation of partially defined concepts. Boundary regions of approximated concepts are changing together with the subjective knowledge on which approximation is based. In Section 2.3 a relationship of this property with the so-called higher order vagueness is pointed out. Approximation spaces are becoming more compound in the case of information granulation or hierarchical learning. Some issues related to such approximation spaces are discussed in Sections 2.4 and 2.5. In particular, the ontology approximation problem is formulated and its importance for compound concept approximation is emphasized. This is a problem of approximation of domain knowledge specified by vague concepts and dependencies between them given together with partial information about the vague concepts. A generalization of mereology due to Leśniewski to rough mereology is outlined in Section 2.6. Rough mereology has been developed for synthesis and analysis of compound objects satisfying a given specification to a satisfactory degree. Finally, in Section 3 some information on the current methods and applications based on rough sets is included together with an example of the challenging research problem concerning approximation of compound concepts.

2. Extensions

The rough set concept can be defined quite generally by means of topological operations, *interior* and *closure*, called *approximations* [52]. It was observed in [44] that the key to the presented approach is provided by the exact mathematical formulation of the concept of approximative (rough) equality of sets in a given approximation space. In [45], an approximation space is represented by the pair (U, R), where U is a universe of objects, and $R \subseteq U \times U$ is an indiscernibility relation defined by an attribute set (i.e., $R = I(A)$ for some attribute set A). In this case R is the equivalence relation. Let $[x]_R$ denote an equivalence class of an element $x \in U$ under the indiscernibility relation R, where $[x]_R = \{y \in U : xRy\}$.

In this context, R-approximations of any set $X \subseteq U$ are based on the exact (crisp) containment of sets. Then set approximations are defined as follows:

- $x \in U$ belongs with certainty to $X \subseteq U$ (i.e., x belongs to the R-lower approximation of X), if $[x]_R \subseteq X$.
- $x \in U$ possibly belongs $X \subseteq U$ (i.e., x belongs to the R-upper approximation of X), if $[x]_R \cap X \neq \emptyset$.
- $x \in U$ belongs with certainty neither to the X nor to $U - X$ (i.e., x belongs to the R-boundary region of X), if $[x]_R \cap (U - X) \neq \emptyset$ and $[x]_R \cap X \neq \emptyset$.

Several generalizations of the above approach have been proposed in the literature (see, e.g., [1,18,41,71,76,83,84,87,93,100]). In particular, in some of these approaches, set inclusion to a degree is used instead of the exact inclusion.

Different aspects of vagueness in the rough set framework are discussed, e.g., in [34,38,39,63,68].

Our knowledge about the approximated concepts is often partial and uncertain [23]. For example, concept approximation should be constructed from examples and counterexamples of objects for the concepts [15]. Hence, concept approximations constructed from a given sample of objects are extended, using inductive reasoning, on objects not yet observed. The rough set approach for dealing with concept approximation under such partial knowledge is presented, e.g., in [76]. Moreover, the concept approximations should be constructed under dynamically changing environments [68,75]. This leads to a more complex situation where the boundary regions are not crisp sets, which is consistent with the postulate of the higher order vagueness considered by philosophers (see, e.g., [26]). It is worthwhile to mention that a rough set approach to the approximation of compound concepts has been developed and at this time no traditional method is able directly to approximate compound concepts [11,92]. The approach is based on hierarchical learning and ontology approximation [8,36,41,70]. Approximation of concepts in distributed environments is discussed in [66]. A survey of algorithmic methods for concept approximation based on rough sets and Boolean reasoning in presented, e.g., in [4,64].

2.1. Generalizations of approximation spaces

Several generalizations of the classical rough set approach based on approximation spaces defined as pairs of the form (U, R), where R is the equivalence relation (called indiscernibility relation) on the set U, have been reported in the literature. Let us mention two of them.

A generalized approximation space[1] can be defined by a tuple $AS = (U, I, v)$ where I is the *uncertainty function* defined on U with values in the powerset $\mathrm{Pow}(U)$ of U ($I(x)$ is the *neighborhood* of x) and v is the *inclusion function* defined on the Cartesian product $\mathrm{Pow}(U) \times \mathrm{Pow}(U)$ with values in the interval $[0, 1]$ measuring the degree of inclusion of sets. The lower AS_* and upper AS^* approximation operations can be defined in AS by

$$AS_*(X) = \{x \in U : v(I(x), X) = 1\}, \quad (1)$$

$$AS^*(X) = \{x \in U : v(I(x), X) > 0\}. \quad (2)$$

[1] Some other generalizations of approximation spaces are also considered in the literature (see, e.g., [31,32,57,65,91,94–97]).

In the standard case $I(x)$ is equal to the equivalence class $B(x)$ of the indiscernibility relation $I(B)$; in case of tolerance (similarity) relation $\tau \subseteq U \times U$ [62] we take $I(x) = \{y \in U : x\tau y\}$, i.e., $I(x)$ is equal to the tolerance class of τ defined by x. The standard inclusion relation ν_{SRI} is defined for $X, Y \subseteq U$ by

$$\nu_{\mathrm{SRI}}(X,Y) = \begin{cases} \frac{card(X\cap Y)}{card(X)}, & \text{if } X \text{ is non-empty,} \\ 1, & \text{otherwise.} \end{cases} \tag{3}$$

For applications it is important to have some constructive definitions of I and ν.

One can consider another way to define $I(x)$. Usually together with AS we consider some set F of formulae describing sets of objects in the universe U of AS defined by semantics $\|\cdot\|_{AS}$, i.e., $\|\alpha\|_{AS} \subseteq U$ for any $\alpha \in F$. Now, one can take the set

$$N_F(x) = \{\alpha \in F : x \in \|\alpha\|_{AS}\}, \tag{4}$$

and $I(x) = \{\|\alpha\|_{AS} : \alpha \in N_F(x)\}$. Hence, more general uncertainty functions having values in Pow(Pow(U)) can be defined and in the consequence different definitions of approximations are considered. For example, one can consider the following definitions of approximation operations in AS:

$$AS_\circ(X) = \{x \in U : \nu(Y,X) = 1 \text{ for some } Y \in I(x)\}, \tag{5}$$
$$AS^\circ(X) = \{x \in U : \nu(Y,X) > 0 \text{ for any } Y \in I(x)\}. \tag{6}$$

There are also different forms of rough inclusion functions. Let us consider two examples.

In the first example of a rough inclusion function, a threshold $t \in (0, 0.5)$ is used to relax the degree of inclusion of sets. The rough inclusion function ν_t is defined by

$$\nu_t(X,Y) = \begin{cases} 1, & \text{if } \nu_{\mathrm{SRI}}(X,Y) \geqslant 1-t, \\ \frac{\nu_{\mathrm{SRI}}(X,Y)-t}{1-2t}, & \text{if } t \leqslant \nu_{\mathrm{SRI}}(X,Y) < 1-t, \\ 0, & \text{if } \nu_{\mathrm{SRI}}(X,Y) \leqslant t. \end{cases} \tag{7}$$

This is an interesting "rough-fuzzy" example because we put the standard rough membership function as an argument into the formula often used for fuzzy membership functions.

One can obtain approximations considered in the variable precision rough set approach (VPRSM) [100] by substituting in (1), (2) the rough inclusion function ν_t defined by (7) instead of ν, assuming that Y is a decision class and $N(x) = B(x)$ for any object x, where B is a given set of attributes.

Another example of application of the standard inclusion was developed by using probabilistic decision functions. For more detail the reader is referred to [79–81].

The rough inclusion relation can be also used for function approximation [76] and relation approximation [89]. In the case of function approximation the inclusion function ν^* for subsets $X, Y \subseteq U \times U$, where $X, Y \subseteq \mathscr{R}$ and $\mathscr{R}$ is the set of reals, is defined by

$$\nu^*(X,Y) = \begin{cases} \frac{card(\pi_1(X\cap Y))}{card(\pi_1(X))}, & \text{if } \pi_1(X) \neq \emptyset, \\ 1, & \text{if } \pi_1(X) = \emptyset, \end{cases} \tag{8}$$

where π_1 is the projection operation on the first coordinate. Assume now, that X is a cube and Y is the graph $G(f)$ of the function $f : \mathscr{R} \to \mathscr{R}$. Then, e.g., X is in the lower approximation of f if the projection on the first coordinate of the intersection $X \cap G(f)$ is equal to the projection of X on the first coordinate. This means that the part of the graph $G(f)$ is "well" included in the box X, i.e., for all arguments that belong to the box projection on the first coordinate the value of f is included in the box X projection on the second coordinate.

The approach based on inclusion functions has been generalized to the *rough mereological approach* [41,56,55,58] (see also Section 2.6). The inclusion relation $x\mu_r y$ with the intended meaning *x is a part of y to a degree at least r* has been taken as the basic notion of the rough mereology being a generalization of the Leśniewski mereology [29,30]. Research on rough mereology has shown importance of another notion, namely *closeness* of compound objects (e.g., concepts). This can be defined by $xcl_{r,r'}y$ if and only if $x\mu_r y$ and $y\mu_{r'}x$.

Rough mereology offers a methodology for synthesis and analysis of objects in a distributed environment of intelligent agents, in particular, for synthesis of objects satisfying a given specification to a satisfactory degree or for control in such a complex environment. Moreover, rough mereology has been used for developing the foundations of the *information granule calculi*, aiming at formalization of the Computing with Words paradigm, recently formulated by Lotfi Zadeh [98]. More complex information granules are defined recursively using already defined information granules and their measures of inclusion and closeness. Information granules can have complex structures like classifiers or approximation spaces. Computations on information granules are performed to discover relevant information granules, e.g., patterns or approximation spaces for compound concept approximations.

Usually there are considered families of approximation spaces labeled by some parameters. By tuning such parameters according to chosen criteria (e.g., minimal description length) one can search for the optimal approximation space for concept description (see, e.g., [4]).

2.2. Concept approximation

In this section, we consider the problem of approximation of concepts over a universe U^{∞} (concepts that are subsets of U^{∞}). We assume that the concepts are perceived only through some subsets of U^{∞}, called samples. This is a typical situation in the machine learning, pattern recognition, or data mining approaches [15,27,33]. We explain the rough set approach to induction of concept approximations using the generalized approximation spaces of the form $AS = (U, I, \nu)$ defined in Section 2.1.

Let $U \subseteq U^{\infty}$ be a finite sample. By Π_U we denote a perception function from $P(U^{\infty})$ into $P(U)$ defined by $\Pi_U(C) = C \cap U$ for any concept $C \subseteq U^{\infty}$. Let $AS = (U, I, \nu)$ be an approximation space over the sample U.

The problem we consider is how to extend the approximations of $\Pi_U(C)$ defined by AS to approximation of C over U^{∞}. We show that the problem can be described as searching for an extension $AS_C = (U^{\infty}, I_C, \nu_C)$ of the approximation space AS, relevant for approximation of C. This requires to show how to extend the inclusion function ν from subsets of U to subsets of U^{∞} that are relevant for the approximation of C. Observe that for the approximation of C it is enough to induce the necessary values of the inclusion function ν_C without knowing the exact value of $I_C(x) \subseteq U^{\infty}$ for $x \in U^{\infty}$.

Let AS be a given approximation space for $\Pi_U(C)$ and let us consider a language L in which the neighborhood $I(x) \subseteq U$ is expressible by a formula $pat(x)$, for any $x \in U$. It means that $I(x) = \|pat(x)\|_U \subseteq U$, where $\|pat(x)\|_U$ denotes the meaning of $pat(x)$ restricted to the sample U. In case of rule based classifiers patterns of the form $pat(x)$ are defined by feature value vectors.

We assume that for any new object $x \in U^{\infty} \setminus U$ we can obtain (e.g., as a result of sensor measurement) a pattern $pat(x) \in L$ with semantics $\|pat(x)\|_{U^{\infty}} \subseteq U^{\infty}$. However, the relationships between information granules over U^{∞} like sets: $\|pat(x)\|_{U^{\infty}}$ and $\|pat(y)\|_{U^{\infty}}$, for different $x, y \in U^{\infty}$, are, in general, known only if they can be expressed by relationships between the restrictions of these sets to the sample U, i.e., between sets $\Pi_U(\|pat(x)\|_{U^{\infty}})$ and $\Pi_U(\|pat(y)\|_{U^{\infty}})$.

The set of patterns $\{pat(x) : x \in U\}$ is usually not relevant for approximation of the concept $C \subseteq U^{\infty}$. Such patterns are too specific or not enough general, and can directly be applied only to a very limited number of new objects. However, by using some generalization strategies, one can search, in a family of patterns definable from $\{pat(x) : x \in U\}$ in L, for such new patterns that are relevant for approximation of concepts over U^{∞}. Let us consider a subset $PATTERNS(AS, L, C) \subseteq L$ chosen as a set of pattern candidates for relevant approximation of a given concept C. For example, in case of rule based classifier one can search for such candidate patterns among sets definable by subsequences of feature value vectors corresponding to objects from the sample U. The set $PATTERNS(AS, L, C)$ can be selected by using some quality measures checked on meanings (semantics) of its elements restricted to the sample U (like the number of examples from the concept $\Pi_U(C)$ and its complement that support a given pattern). Then, on the basis of properties of sets definable by these patterns over U we induce approximate values of the inclusion function ν_C on subsets of U^{∞} definable by any of such pattern and the concept C.

Next, we induce the value of ν_C on pairs (X, Y) where $X \subseteq U^{\infty}$ is definable by a pattern from $\{pat(x) : x \in U^{\infty}\}$ and $Y \subseteq U^{\infty}$ is definable by a pattern from $PATTERNS(AS, L, C)$.

Finally, for any object $x \in U^{\infty} \setminus U$ we induce the approximation of the degree $\nu_C(\|pat(x)\|_{U^{\infty}}, C)$ applying a conflict resolution strategy *Conflict_res* (a voting strategy, in case of rule based classifiers) to two families of degrees:

$$\{\nu_C(\|pat(x)\|_{U^{\infty}}, \|pat\|_{U^{\infty}}) : pat \in PATTERNS(AS, L, C)\}, \tag{9}$$

$$\{\nu_C(\|pat\|_{U^{\infty}}, C) : pat \in PATTERNS(AS, L, C)\}. \tag{10}$$

Values of the inclusion function for the remaining subsets of U^{∞} can be chosen in any way – they do not have any impact on the approximations of C. Moreover, observe that for the approximation of C we do not need to know the exact values of uncertainty function I_C – it is enough to induce the values of the inclusion function ν_C. Observe that the defined extension ν_C of ν to some subsets of U^{∞} makes it possible to define an approximation of the concept C in a new approximation space AS_C.

Observe that one can also follow principles of Bayesian reasoning and use degrees of ν_C to approximate C (see, e.g., [46,82,85]).

In this way, the rough set approach to induction of concept approximations can be explained as a process of inducing a relevant approximation space.

2.3. Higher order vagueness

In [26], it is stressed that vague concepts should have non-crisp boundaries. In the definition presented in [47], the notion of boundary region is defined as a crisp set $BN_B(X)$. However, let us observe that this definition is relative to the subjective knowledge expressed by attributes from B. Different sources of information may use different sets of attributes for concept approximation. Hence, the boundary region can change when we consider these different views. Another aspect is discussed in [68,75] where it is assumed that information about concepts is incomplete, e.g., the concepts are given only on samples (see, e.g., [15,27,35]). From [68,75] it follows that vague concepts cannot be approximated with satisfactory quality by *static* constructs such as induced membership inclusion functions, approximations or models derived, e.g., from a sample. Understanding of vague concepts can be only realized in a process in which the induced models are adaptively matching the concepts in a dynamically changing environment. This conclusion seems to have important consequences for further development of rough set theory in combination with fuzzy sets and other soft computing paradigms for adaptive approximate reasoning.

2.4. Information granulation

Information granulation can be viewed as a human way of achieving data compression and it plays a key role in the implementation of the strategy of divide-and-conquer in human problem-solving [98]. Objects obtained as the result of granulation are information granules. Examples of elementary information granules are indiscernibility or tolerance (similarity) classes (see, e.g., [47]). In reasoning about data and knowledge under uncertainty and imprecision many other more compound information granules are used (see, e.g., [61,59,65,72,73]). Examples of such granules are decision rules, sets of decision rules or classifiers. More compound information granules are defined by means of less compound ones. Note that inclusion or closeness measures between information granules should be considered rather than their strict equality. Such measures are also defined recursively for information granules.

Let us discuss shortly an example of information granulation in the process of modeling patterns for compound concept approximation (see, e.g., [6–10,36,78,90]). We start from a generalization of information systems. For any attribute $a \in A$ of an information system (U, A) we consider together with the value set V_a of a a relational structure $\mathcal{R}_a$ over the universe V_a (see, e.g., [77]). We also consider a language $\mathcal{L}_a$ of formulas (of the same relational signature as $\mathcal{R}_a$). Such formulas interpreted over $\mathcal{R}_a$ define subsets of Cartesian products of V_a. For example, any formula α with one free variable defines a subset $\|\alpha\|_{\mathcal{R}_a}$ of V_a. Let us observe that the relational structure $\mathcal{R}_a$ (without functions) induces a relational structure over U. Indeed, for any k-ary relation r from $\mathcal{R}_a$ one can define a k-ary relation $g_a \subseteq U^k$ by $(x_1, \ldots, x_k) \in g_a$ if and only if $(a(x_1), \ldots, a(x_k)) \in r$ for any $(x_1, \ldots, x_k) \in U^k$. Hence, one can consider any formula from $\mathcal{L}_a$ as a constructive method of defining a subset of the universe U with a structure induced by $\mathcal{R}_a$. Any such a structure is a new information granule. On the next level of hierarchical modeling, i.e., in constructing new information systems we use such structures as objects and attributes are properties of such structures. Next, one can consider similarity between new constructed objects and then their similarity neighborhoods will correspond to clusters of relational structures. This process is usually more complex. This is because instead of relational structure $\mathcal{R}_a$ we usually consider a fusion of relational structures corresponding to some attributes from A. The fusion makes it possible to describe constraints that should hold between parts obtained by composition from less compound parts. Examples of relational structures can be defined by indiscernibility, similarity, intervals obtained in discretization or symbolic value grouping, preference or spatio-temporal relations (see, e.g., [18,27,71]). One can see that parameters to be tuned in searching for relevant[2] patterns over new information systems are, among others, relational structures over value sets, the language of formulas defining parts, and constraints.

2.5. Ontological framework for approximation

In a number of papers (see, e.g., [40,74]), the problem of ontology approximation has been discussed together with possible applications to approximation of compound concepts or to knowledge transfer (see, e.g., [5,37,69,67,74]).

In the ontology [88] (vague) concepts and local dependencies between them are specified. Global dependencies can be derived from local dependencies. Such derivations can be used as hints in searching for relevant compound patterns (information granules) in approximation of more compound concepts from the ontology. The ontology approximation problem is one of the fundamental problems related to approximate reasoning in distributed environments. One should construct (in a given language that is different from the ontology specification language) not only approximations of concepts from ontology but also vague dependencies specified in the ontology. It is worthwhile to mention that an ontology approximation should be induced on the basis of incomplete information about concepts and dependencies specified in the ontology. Information granule calculi based on rough sets have been proposed as tools making it possible to solve this problem. Vague dependencies have vague concepts in premisses and conclusions. The approach to approximation of vague dependencies based only on degrees of closeness of concepts from dependencies and their approximations (classifiers) is not satisfactory for approximate reasoning. Hence, more advanced approach should be developed. Approximation of any vague dependency is a method which allows for any object to compute the arguments "for" and "against" its membership to the dependency conclusion on the basis of the analogous arguments relative to the dependency premisses. Any argument is a compound information granule (compound pattern). Arguments are fused by local schemes (production rules) discovered from data. Further fusions are possible through composition of local schemes, called approximate reasoning schemes (AR schemes) (see, e.g., [9,41,59]). To estimate the degree to which (at least) an object belongs to concepts from ontology the arguments "for" and "against" those concepts are collected and next a conflict resolution strategy is applied to them to predict the degree.

2.6. Mereology and rough mereology

This section introduces some basic concepts of rough mereology (see, e.g., [53,54,56,59–61]).

Exact and rough concepts can be characterized by a new notion of an element, alien to naive set theory in which this theory has been coded until now. For an information system $\mathcal{A} = (U, A)$, and a set B of attributes, the mereological element $el_B^{\mathcal{A}}$ is defined by letting

$$x el_B^{\mathcal{A}} X \text{ if and only if } B(x) \subseteq X. \tag{11}$$

Then, a concept X is B-exact if and only if either $x el_B^{\mathcal{A}} X$ or $x el_B^{\mathcal{A}} U \setminus X$ for each $x \in U$, and the concept X is B-rough if and only if for some $x \in U$ neither $x el_B^{\mathcal{A}} X$ nor $x el_B^{\mathcal{A}} U \setminus X$.

[2] For target concept approximation.

Thus, the characterization of the dichotomy exact-rough cannot be done by means of the element notion of naive set theory, but requires the notion of containment ($\subseteq$), i.e., a notion of mereological element.

The Leśniewski Mereology (theory of parts) is based on the notion of a part [29,30]. The relation π of part on the collection U of objects satisfies

1. if $x\pi y$ then not $y\pi x$; (12)
2. if $x\pi y$ and $y\pi z$ then $x\pi z$. (13)

The notion of mereological element el_π is introduced as

$$x el_\pi y \text{ if and only if } x\pi y \text{ or } x = y. \tag{14}$$

In particular, the relation of proper inclusion $\subset$ is a part relation π on any non-empty collection of sets, with the element relation $el_\pi = \subseteq$.

Formulas expressing, e.g., rough membership, quality of decision rule, quality of approximations can be traced back to a common root, i.e., $v(X, Y)$ defined by Eq. (3). The value $v(X, Y)$ defines the degree of *partial containment* of X into Y and naturally refers to the Leśniewski Mereology. An abstract formulation of this idea in [56] connects the mereological notion of element el_π with the partial inclusion by introducing a *rough inclusion* as a relation $v \subseteq U \times U \times [0,1]$ on a collection of pairs of objects in U endowed with part π relation, and such that

1. $v(x, y, 1)$ if and only if $x el_\pi y$; (15)
2. if $v(x, y, 1)$ then (if $v(z, x, r)$ then $v(z, y, r)$); (16)
3. if $v(z, x, r)$ and $s < r$ then $v(z, x, s)$. (17)

Implementation of this idea in information systems can be based on *Archimedean* t-norms [56]; each such norm T is represented as $T(r, s) = g(f(r) + f(s))$ with f, g pseudo-inverses to each other, continuous and decreasing on $[0, 1]$. Letting for (U, A) and $x, y \in U$

$$DIS(x, y) = \{a \in A : a(x) \neq a(y)\}, \tag{18}$$

and

$$v(x, y, r) \text{ if and only if } g\left(\frac{card(DIS(x, y))}{card(A)}\right) \geqslant r, \tag{19}$$

v defines a rough inclusion that satisfies additionally the transitivity rule

$$\frac{v(x, y, r), v(y, z, s)}{v(x, z, T(r, s))}. \tag{20}$$

Simple examples here are: the Menger rough inclusion in the case $f(r) = -\ln r$, $g(s) = \mathrm{e}^{-s}$ yields $v(x, y, r)$ if and only if $\mathrm{e}^{-\frac{card(DIS(x,y))}{card(A)}} \geqslant r$ and it satisfies the transitivity rule:

$$\frac{v(x, y, r), v(y, z, s)}{v(x, y, r \cdot s)}, \tag{21}$$

i.e., the t-norm T is the Menger (product) t-norm $r \cdot s$, and, the Łukasiewicz rough inclusion with $f(x) = 1 - x = g(x)$ yielding $v(x, y, r)$ if and only if $1 - \frac{card(DIS(x,y))}{card(A)} \geqslant r$ with the transitivity rule:

$$\frac{v(x, y, r), v(y, z, s)}{v(x, y, \max\{0, r + s - 1\})}, \tag{22}$$

i.e., with the Łukasiewicz t-norm.

Rough inclusions [56] can be used in *granulation of knowledge* [98]. Granules of knowledge are constructed as aggregates of indiscernibility classes close enough with respect to a chosen measure of closeness. In a nutshell, a granule $g_r(x)$ about x of radius r can be defined as the aggregate of all y with $v(y, x, r)$. The aggregating mechanism can be based on the class operator of mereology (cf. rough mereology [56]) or on set theoretic operations of union.

Rough mereology [56] combines rough inclusions with methods of mereology. It employs the operator of mereological class that makes collections of objects into objects. The class operator *Cls* satisfies the requirements, with any non-empty collection M of objects made into the object $Cls(M)$

$$\text{if } x \in M \text{ then } x el_\pi Cls(M), \tag{23}$$

$$\text{if } x el_\pi Cls(M) \text{ then there exist } y, z \text{ such that } y el_\pi x, y el_\pi z, z \in M. \tag{24}$$

In case of the part relation $\subset$ on a collection of sets, the class $Cls(M)$ of a non-empty collection M is the union $\cup M$.

Granulation by means of the class operator *Cls* consists in forming the granule $g_r(x)$ as the class $Cls(y : v(y, x, r))$. One obtains a granule family with regular properties (see [93]).

3. Rough sets: a challenge

There are many real-life problems that are still hard to solve using the existing methodologies and technologies. Among such problems are, e.g., classification of medical images, control of autonomous systems like unmanned aerial vehicles or robots, problems related to monitoring or rescue tasks in multi-agent systems. All of these problems are closely related to intelligent systems that are more and more widely applied in different real-life projects.

One of the main challenges in developing intelligent systems is discovering methods for approximate reasoning from measurements to perception, i.e., deriving from concepts resulting from sensor measurements concepts enunciated in natural language that express perception [98].

Nowadays, new emerging computing paradigms are investigated attempting to make progress in solving problems related to this challenge. Further progress depends on a successful cooperation of specialists from different scientific disciplines such as mathematics, computer science, artificial intelligence, biology, physics, chemistry, bioinformatics, medicine, neuroscience, linguistics, psychology, sociology. In particular, different aspects of reasoning from measurements to perception are investigated in psychology [3,24], neuroscience [51], layered learning [90], mathematics of learning [51], machine learning, pattern recognition [15], data mining [27] and also by researchers working on recently emerged computing paradigms such as computing with words and perception [98], granular computing [41], rough sets, rough-mereology, and rough-neural computing [41].

One of the main problems investigated in machine learning, pattern recognition [15] and data mining [27] is concept approximation. It is necessary to induce approximations of concepts (models of concepts) from available experimental data. The data models developed so far in such areas like statistical learning, machine learning, pattern recognition are not satisfactory for approximation of compound concepts resulting in the perception process. Researchers from the different areas have recognized the necessity to work on new methods for concept approximation (see, e.g., [11,92]). The main reason is that these compound concepts are, in a sense, too far from measurements which makes the searching for relevant (for their approximation) features infeasible in a huge space. There are several research directions aiming at overcoming this difficulty. One of them is based on the interdisciplinary research where the results concerning perception in psychology or neuroscience are used to help to deal with compound concepts (see, e.g., [15]). There is a great effort in neuroscience towards understanding the hierarchical structures of neural networks in living organisms [14,51]. Also mathematicians are recognizing problems of learning as the main problem of the current century [51]. The problems discussed so far are also closely related to complex system modeling. In such systems again the problem of concept approximation and reasoning about perceptions using concept approximations is one of the challenges nowadays. One should take into account that modeling complex phenomena entails the use of local models (captured by local agents, if one would like to use the multi-agent terminology [25,99]) that next should be fused. This process involves the negotiations between agents [25] to resolve contradictions and conflicts in local modeling. This kind of modeling will become more and more important in solving complex real-life problems which we are unable to model using traditional analytical approaches. The latter approaches lead to exact models. However, the necessary assumptions used to develop them are causing the resulting solutions to be too far from reality to be accepted. New methods or even a new science should be developed for such modeling [16].

One of the possible solutions in searching for methods for compound concept approximations is the layered learning idea [90]. Inducing concept approximation should be developed hierarchically starting from concepts close to sensor measurements to compound target concepts related to perception. This general idea can be realized using additional domain knowledge represented in natural language. For example, one can use principles of behavior on the roads, expressed in natural language, trying to estimate, from recordings (made, e.g., by camera and other sensors) of situations on the road, if the current situation on the road is safe or not. To solve such a problem one should develop methods for concept approximations together with methods aiming at approximation of reasoning schemes (over such concepts) expressed in natural language. Foundations of such an approach are based on rough set theory [45] and its extension rough mereology [41,55,56,58], both discovered in Poland.

Objects we are dealing with are information granules (see Section 2.4). Such granules are obtained as the result of information granulation [98].

Computing with Words and Perception "derives from the fact that it opens the door to computation and reasoning with information which is perception-rather than measurement-based. Perceptions play a key role in human cognition, and underlie the remarkable human capability to perform a wide variety of physical and mental tasks without any measurements and any computations. Everyday examples of such tasks are driving a car in city traffic, playing tennis and summarizing a story" [98].

The rough mereological approach [41,55,56,58] is based on calculi of information granules for constructing compound concept approximations. Constructions of information granules should be robust with respect to their input information granule deviations. In this way also a granulation of information granule constructions is considered. As the result we obtain the so-called AR schemes (AR networks) [41,55,56,58]. AR schemes can be interpreted as complex patterns [27]. Searching methods for such patterns relevant for a given target concept have been developed [41]. Methods for deriving relevant AR schemes are of high computational complexity. The complexity can be substantially reduced by using domain knowledge. In such a case AR schemes are derived along reasoning schemes in natural language that are retrieved from domain knowledge. Developing methods for deriving such AR schemes is one of the main goals of our projects.

The outlined research directions create foundations toward understanding the nature of reasoning from measurements to perception. These foundations are crucial for constructing intelligent systems for many real-life projects.

4. Conclusions

The rough set concept has led to its various generalizations. Some of them have been discussed in the article. Among extensions not discussed in this paper is the rough set approach to multi-criteria decision making (see, e.g., [17–22,49,50,86]).

Recently, it has been shown that the rough set approach can be used for synthesis and analysis of concept approximations in the distributed environment of intelligent agents. We outlined the rough mereological approach and its applications in calculi of information granules for synthesis of information granules satisfying a given specification to a satisfactory degree. Finally, we have discussed a challenge for research on rough sets related to approximate reasoning from measurements to perception.

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