Approximation Spaces and Information Granulation

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Abstract. We discuss approximation spaces in the granular computing framework. Such approximation spaces generalise the approaches to concept approximation existing in rough set theory. Approximation spaces are constructed as higher level information granules and are obtained as the result of complex modelling. We present illustrative examples of modelling approximation spaces including approximation spaces for function approximation, inducing concept approximation, and some other information granule approximations. In modelling of such approximation spaces we use an important assumption that not only objects but also more complex information granules involved in approximations are perceived using only partial information about them.

1 Introduction

The rough set approach is based on the approximation space concept. Approximation spaces for information systems [1] are defined by partitions or coverings defined by attributes of pattern space. One can distinguish two basic components in approximation spaces: uncertainty function and inclusion function [2]. This approach has been generalised to the rough mereological approach (see, e.g., [3–5]). The existing approaches are based on observation that the objects are perceived by information about them and due to the incomplete information they can be indiscernible. Hence, with each object one can associate an indiscernibility class, called also (indiscernibility) neighbourhood [6]. In the consequence, testing if given object belongs to a set is substituted by checking a degree to which its neighbourhood is included into the set. Such an approach covers several generalizations of set approximations like these based on tolerance relation or variable precision rough set model [7].

In real-life applications approximation spaces are complex information granules that are not given directly with data but they should be discovered from available data and domain knowledge by some searching strategies (see, e.g., [5, 8]). In the paper we

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present a general approach to approximation spaces based on granular computing. We show that the existing approaches to approximations in rough sets are particular cases of our approach. Illustrative examples include approximation spaces with complex neighbourhoods, approximation spaces for function approximation and for inducing concept approximations. Some other aspects of information granule construction, relevant for approximation spaces, are also presented. Furthermore, we discuss one more aspect of approximation spaces based on the observation that the definition of approximations does not depend only on perception of partial information about objects but also on perception of more complex information granules.

The presented approach can be interpreted in multi-agent setting [9, 5]. Each agent is equipped with its own relational structure and approximation spaces located in input ports. The approximation spaces are used for filtering (approximating) information granules sent by other agents. Such agents are performing operations on approximated information granules and sending the results to other agents, checking relationships between approximated information granules, or using such granules in negotiations with other agents. Parameters of approximation spaces are analogous to weights in classical neuron models. Agents are performing operations on information granules rather than on numbers. This analogy has been used as a starting point for the rough-neural computing paradigm [10] of computing with words [11].

2 Granule Approximation Spaces

Using the granular computing approach (see, e.g., [5]) one can generalise the approximation operations for sets of objects, known in rough set theory, to arbitrary information granules. The basic concept is the rough inclusion function ν [3–5].

First, let us recall the definition of information granule system [5].

Definition 1. *An* information granule system *is any tuple* $GS = (E, O, G, \nu)$ *where* E *is a set of elements called* elementary information granules*,* O *is a set of (partial)* operations *on information granules,* G *is a set of information granules constructed from* E *using operations from* O*, and* $\nu : G \times G \longrightarrow [0, 1]$ *is a (partial) function called* rough inclusion.

The main interpretation of rough inclusion is to measure the inclusion degree of one granule in another. In the paper we use the following notation: $\nu_p(g, g')$ denotes that $\nu(g, g') \geq p$ holds; $Gran(GS) = G$; $\mathcal{G}$ denotes a given family of granule systems. $P(X)$ denotes the set of all subsets of X, $P^1(X) = P(X), P^{k+1}(X) = P(P^k(X))$ for $k \geq 1$.

We begin with the general definition of approximation space in the context of a family of information granule systems.

Definition 2. *Let* $\mathcal{G}$ *be a family of information granule systems. A* granule approximation space *with respect to* $\mathcal{G}$ *is any tuple*

$$\mathcal{AS}_{\mathcal{G}} = (GS, G, Tr), \tag{1}$$

where GS *is a selected (initial) granule system from* $\mathcal{G}$*;* $G \subseteq Gran(GS)$ *is some collection of granules; transition relation* Tr *is a binary relation on information granule systems from* $\mathcal{G}$*, i.e.,* $Tr \subseteq \mathcal{G} \times \mathcal{G}$*.*

Let $GS \in \mathcal{G}$. By $Tr[GS]$ we denote the set of all information granule systems reachable from GS:

$$Tr[GS] = \{GS' \in \mathcal{G} : GS\, Tr^*\, GS'\}, \tag{2}$$

where Tr^* is the reflexive and transitive closure of the relation Tr. By $Tr[GS, G]$ we denote the set of all Tr-terminal granule systems reachable from GS that consist of information granules from G:

$$Tr[GS, G] = \{GS' : (GS, GS') \in Tr^*, G \subseteq Gran(GS'), \text{and } Tr[GS'] = \{GS'\}\}. \tag{3}$$

The elements of $Tr[GS, G]$ are called the candidate granule systems for approximation of information granules from G, generated by Tr from GS (G-candidates, for short). For any system $GS^* \in Tr[GS, G]$ we define approximations of granules from G by information granules from $Gran(GS^*) \setminus G$. Searching for granule systems from $Tr[GS, G]$ that are relevant for approximation of given information granules is one of the most important tasks in granular computing.

By using granule approximation space $\mathcal{AS}_{\mathcal{G}} = (GS, G, Tr)$, for a family of granule systems $\mathcal{G}$, we can define approximation of a given granule $g \in G$ in terms of its lower and upper approximations[1]. We assume that there is additionally given "make granule" operation $\oplus : P(G^*) \longrightarrow G^*$, where $G^* = Gran(GS^*)$, for any $GS^* \in Tr(GS, G)$, that constructs a single granule from a set of granules. A typical example of $\oplus$ is set theoretical union, however, it can be also realised by a complex classifier. The granule approximation is thus defined as follows:

Definition 3. *Let* $0 \leq p < q \leq 1$, $\mathcal{AS} = (GS, G, Tr)$ *be a granule approximation space, and* $GS^* \in Tr[GS, G]$. *The* $(\mathcal{AS}, GS^*, \oplus, q)$-lower approximation *of* $g \in G$ *is defined by*

$$LOW(\mathcal{AS}, GS^*, \oplus, q)(g) = \oplus(\{g' \in Gran(GS^*) \setminus G : \nu^*(g', g) \geq q\}) \tag{4}$$

where ν^* *denotes the rough inclusion of* GS^*. *The* $(\mathcal{AS}, GS^*, \oplus, p)$-upper approximation *of* g *is defined by*

$$UPP(\mathcal{AS}, GS^*, \oplus, p)(g) = \oplus(\{g' \in Gran(GS^*) \setminus G : \nu^*(g', g) > p\}) \tag{5}$$

where ν^* *denotes the rough inclusion of* GS^*.

The numbers p, q can be interpreted as inclusion degrees that make it possible to control the size of the lower and upper approximations. In case when $p = 0$ and $q = 1$ we have the case of full inclusion (lower approximation) and any non-zero inclusion (upper approximation).

One can search for optimal approximations of granules from G defined by $GS^* \in Tr[GS, G]$ using some optimisation criteria or one can search for relevant fusion of approximations of granules from G defined by $GS^* \in Tr[GS, G]$ [2].

In the following sections we discuss illustrative examples showing that the above scheme generalises several approaches to approximation spaces and set approximations. In particular, we include examples of information granules G and their structures, the rough inclusion ν as well as the operation $\oplus$.

[1] If there is no contradiction we use $\mathcal{AS}$ instead of $\mathcal{AS}_{\mathcal{G}}$.

[2] This problem will be investigated elsewhere.

3 Approximation Spaces

Let us recall the definition of an approximation space from [2, 1]. For simplicity of reasoning we omit parameters that label components of approximation spaces.

Definition 4. *An approximation space is a system* $AS = (U, I, \nu)$*, where*

- U *is a non-empty finite set of objects,*
- $I : U \to P(U)$ *is an uncertainty function, such that* $x \in I(x)$ *for any* $x \in U$*,*
- $\nu : P(U) \times P(U) \to [0, 1]$ *is a rough inclusion function.*

A set $X \subseteq U$ is *definable in* AS if and only if it is a union of some values of the uncertainty function. The standard rough inclusion function ν_{SRI} defines the degree of inclusion between two subsets of U by

$$\nu_{SRI}(X, Y) = \begin{cases} \frac{card(X \cap Y)}{card(X)} & \text{if } X \neq \emptyset \\ 1 & \text{if } X = \emptyset. \end{cases} \tag{6}$$

The lower and the upper approximations of subsets of U are defined as follows.

Definition 5. *For any approximation space* $AS = (U, I, \nu)$, $0 \leq p < q \leq 1$*, and any subset* $X \subseteq U$ *the* q-lower *and the* p-upper approximation *of* X *in* AS *is defined, respectively, by*

$$LOW_q(AS, X) = \{x \in U : \nu(I(x), X) \geq q\}, \tag{7}$$

$$UPP_p(AS, X) = \{x \in U : \nu(I(x), X) > p\}. \tag{8}$$

Approximation spaces can be constructed directly from information systems or information systems enriched by some similarity relations on attribute value vectors. The above definition generalises several approaches existing in the literature like these based on equivalence or tolerance indiscernibility relation as well as those based on exact inclusion of indiscernibility classes into concepts [1, 7].

Let us observe that the above definition of approximations is a special case of Definition 3. The granule approximation space $\mathcal{AS} = (GS, G, Tr)$ corresponding to $AS = (U, I, \nu)$ can be defined by

1. GS consisting of information granules being subsets of U (in particular, containing neighbourhoods that are values of the uncertainty function I) and of rough inclusion $\nu = \nu_{SRI}$.
2. $G = P(U)$.
3. $Tr[GS, G]$ consisting of exactly two systems: GS and GS^* such that
   - $Gran(GS^*) = G \cup \{(x, I(x)) : x \in U\}$;
   - the rough inclusion ν is extended by $\nu((x, I(x)), X) = \nu(I(x), X)$ for $x \in U$ and $X \subseteq U$.

Suppose the "make granule" operation $\oplus$ is defined by

$$\oplus(\{(x,\cdot) : x \in Z\}) = Z \text{ for any } Z \subseteq U.$$

Then for the approximation space AS and granule approximation space $\mathcal{AS}$ we have the following:

Proposition 1. *Let* $0 \leq p < q \leq 1$. *For any* $X \in P(U)$ *we have:*

$$LOW_q\,(AS, X) = LOW(\mathcal{AS}, GS^*, \oplus, q)(X), \tag{9}$$

$$UPP_p\,(AS, X) = UPP(\mathcal{AS}, GS^*, \oplus, p)(X). \tag{10}$$

4 Approximation Spaces with Complex Neighbourhoods

In this section we present approximation spaces that have more complex uncertainty functions. Such functions define complex neighbourhoods of objects, e.g., families of sets. This aspect is very important from the point of view of complex concepts approximation. A special case of complex uncertainty functions is such with values in $P^2(U)$, i.e., $I : U \longrightarrow P^2(U)$. Such uncertainty functions appear, e.g., in case of the similarity based rough set approach. One can define $I(x)$ to be a family of all maximal cliques defined by the similarity relation which x belongs to (see Section 7).

We obtain the following definition of approximation space:

Definition 6. *A k-th order approximation space is any tuple* $AS = \left(U, I^k, \nu\right)$, *where*

- U *is a non-empty finite set of objects,*
- $I^k : U \rightarrow P^k\,(U)$ *is an uncertainty function,*
- $\nu : P\,(U) \times P\,(U) \rightarrow [0,1]$ *is a rough inclusion function.*

Let us note that up to the above definition there can be given different uncertainty functions for different levels of granulation. The inclusion function can be also defined in this way, however, in most cases we induce it recursively from ν. For example, in case of set approximation by means of given approximation space $AS = \left(U, I^k, \nu\right)$ we are interested in inclusion function $\nu^k : P^k\,(U) \times P\,(U) \rightarrow [0,1]$, defined recursively by the corresponding relation ν_p^k, for $X \subseteq U$ and $\mathcal{Y} \subseteq P^k(U)$, as follows

- $\nu_p^k(\mathcal{Y}, X)$ iff $\exists Y \in \mathcal{Y}\ \ \nu_p^{k-1}(Y, X)$,
- $\nu_p^1(Y, X)$ iff $\nu_p(Y, X)$.

The definition of set approximations for the k-th order approximation spaces depends on the way the values of uncertainty function are perceived. To illustrate this point of view we consider the following two examples: the complete perception of neighbourhoods and the perception defined by the intersection of the family $I(x)$. In the former case we consider a new definition of set approximations.

Definition 7. *Let* $0 \leq p < q \leq 1$. *For any k-th order approximation space* $AS = \left(U, I^k, \nu\right)$, ν^k *induced from* ν, *and any subset* $X \subseteq U$ *the* q-lower *and the* p-upper approximation *of* X *in* AS *is defined, respectively, by*

$$LOW_q(AS, X) = \left\{x \in U : \nu^k\left(I^k(x), X\right) \geq q\right\}, \tag{11}$$

$$UPP_p(AS, X) = \left\{x \in U : \nu^k\left(I^k(x), X\right) > p\right\}. \tag{12}$$

We can observe that the approximation operations for these two cases are, in general, different.

Proposition 2. *Let us denote by $AS^{\cap} = (U, I^{\cap}, \nu)$ the approximation space obtained from the second order approximation space $AS = \left(U, I^2, \nu\right)$ assuming $I^{\cap}(x) = \cap I^2(x)$ for $x \in U$. We also assume that $x \in Y$ for any $Y \in I^2(x)$ and $x \in U$. Then, for $X \subseteq U$ we have*

$$LOW(AS, X) \subseteq LOW(AS^{\cap}, X) \subseteq X \subseteq UPP(AS^{\cap}, X) \subseteq UPP(AS, X). \tag{13}$$

One can check (in an analogous way as in Section 3) that the above definition of approximations is a special case of Definition 3.

5 Relation and Function Approximation

One can directly apply the definition of set approximation to relations. For simplicity, but without loss of generality, we consider binary relations only. Let $R \subseteq U \times U$ be a binary relation. We can consider approximation of R by an approximation space $AS = (U \times U, I, \nu)$ in an analogous way as in Definition 5:

$$LOW_q(AS, R) = \{(x, y) \in U \times U : \nu(I(x, y)), X) \geq q\}, \tag{14}$$

$$UPP_p(AS, R) = \{(x, y) \in U \times U : \nu(I(x, y), X) > p\}, \tag{15}$$

for $0 \leq p < q \leq 1$. This definition can be also easily extended to the case of complex uncertainty function as in Definition 7. However, the main problem is how to construct relevant approximation spaces, i.e., how to define uncertainty and inclusion functions. One can consider, e.g., uncertainty function $I(x, y) = I(x) \times I(y)$ (assuming that one dimensional uncertainty function is given) and the standard inclusion, i.e., $\nu = \nu_{SRI}$.

Now, let us consider an approximation space $AS = (U, I, \nu)$ and a function $f : Dom \longrightarrow U$, where $Dom \subseteq U$. By $Graph(f)$ we denote the set $\{(x, f(x)) : x \in Dom\}$. We can easily see that if we apply the above definition of relation approximation to f (it is a special case of relation) then the lower approximation is almost always empty. Thus, we have to construct the relevant approximation space $AS^* = (U \times U, I^*, \nu^*)$ in different way, e.g., by extending the uncertainty function as well as the inclusion function on subsets of $U \times U$. We assume that the value $I^*(x, y)$ of the uncertainty function, called the neighbourhood (or the window) of (x, y), for $(x, y) \in U \times U$, is defined by

$$I^*(x, y) = I(x) \times I(y). \tag{16}$$

Next, we should decide how to define values of the inclusion function on pairs $(I^*(x, y), Graph(f))$, i.e., how to define the degree r to which the intersection $I^*(x, y) \cap Graph(f)$ is included into $Graph(f)$.

One can consider a ratio

$$r = \frac{card(\{x \in I(x) \cap Dom : f(x) \in I(y)\})}{card(I(x))}, \tag{17}$$

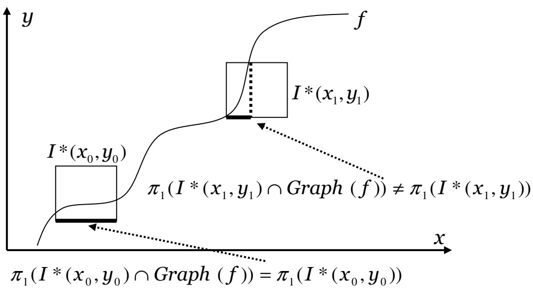

Fig. 1. Function approximation

i.e., the ratio of the number of all objects from $I(x) \cap Dom$ (if any) on which f takes a value from $I(y)$ to the number of objects in $I(x)$.

If $r = 1$ then (x, y) defining the window $I^*(x, y)$ is in the lower approximation of $Graph(f)$; if $0 < r \leq 1$ then (x, y) defining the window $I^*(x, y)$ is in the upper approximation of $Graph(f)$.

Using the above intuition, we assume that the inclusion holds to degree one if the domain of $Graph(f)$ restricted to $I(x)$ is equal to $I(x)$. This can be formally defined by the following condition:

$$\pi_1(I^*(x, y) \cap Graph(f)) = \pi_1(I^*(x, y)), \tag{18}$$

where π_1 denotes the projection on the first coordinate. It is equivalent to

$$\forall x' \in I(x)\ f(x') \in I(y). \tag{19}$$

Thus, the inclusion function ν^* for subsets $X, Y \subseteq U \times U$ is defined by

$$\nu^*(X, Y) = \begin{cases} \frac{card(\pi_1(X \cap Y))}{card(\pi_1(X))} & \text{if } \pi_1(X) \neq \emptyset \\ 1 & \text{if } \pi_1(X) = \emptyset. \end{cases} \tag{20}$$

Hence, the relevant inclusion function in approximation spaces for function approximations is such a function that does not measure the degree of inclusion of its arguments but their perceptions, represented in the above example by projections of corresponding sets. One can choose another definition, e.g., based on the density of pixels (in case of images) in a window that are matched by the function graph.

Proposition 3. *Let* $AS^* = (U \times U, I^*, \nu^*)$ *be an approximation space with* I^*, ν^* *defined by (16), (20), respectively, and let* $f : Dom \longrightarrow U$ *where* $Dom \subseteq U$. *Then*

1. $(x, y) \in LOW_1(AS^*, Graph(f))$ iff $\forall x' \in I(x)\ f(x') \in I(y)$.
2. $(x, y) \in UPP_0(AS^*, Graph(f))$ iff $\exists x' \in I(x)\ f(x') \in I(y)$.

In case of arbitrary parameters p, q satisfying $0 \leq p < q \leq 1$ we have

Proposition 4. *Let* $AS^* = (U \times U, I^*, \nu^*)$ *be an approximation space with* I^*, ν^* *defined by* (16), (20), *respectively, and let* $f : Dom \longrightarrow U$ *where* $Dom \subseteq U$. *Then*

1. $(x, y) \in LOW_q(AS^*, Graph(f))$ iff $card(\{x' \in I(x) : f(x') \in I(y)\}) \geq q \cdot card(I(x))$.
2. $(x, y) \in UPP_p(AS^*, Graph(f))$ iff $card(\{x' \in I(x) : f(x') \in I(y)\}) > p \cdot card(I(x))$.

In our example we define the inclusion degree between two subsets of Cartesian product using, in a sense, the inclusion degree between their projections. Hence, subsets of Cartesian products are perceived by projections.

Again, one can consider the definition of approximation space for function approximation as a special case of the granule approximation space introduced in Definition 2 with the non standard rough inclusion introduced in this section.

6 Concept Approximation

In this section we consider the problem of approximation of concepts over a universe U^∞ (concepts that are subsets of U^∞). We assume that the concepts are perceived only through some subsets of U^∞, called samples. This is a typical situation in the machine learning, pattern recognition, or data mining approaches [12]. In this section we explain the rough set approach to induction of concept approximations.

Let $U \subseteq U^\infty$ be a finite sample. By Π_U we denote a perception function from $P(U^\infty)$ into $P(U)$ defined by $\Pi_U(C) = C \cap U$ for any concept $C \subseteq U^\infty$. Let $AS = (U, I, \nu)$ be an approximation space over the sample U.[3] The problem we consider is how to extend the approximations of $\Pi_U(C)$ defined by AS to approximation of C over U^∞. We show that the problem can be described as searching for an extension $AS_C = (U^\infty, I_C, \nu_C)$ of the approximation space AS, relevant for approximation of C. This requires to show how to extend the inclusion function ν from U to relevant subsets of U^∞ that are suitable for the approximation of C. Observe (cf. Definition 5) that for the approximation of C it is enough to induce the necessary values of the inclusion function ν_C without knowing the exact value of $I_C(x) \subseteq U^\infty$ for $x \in U^\infty$.

Let AS be a given approximation space for $\Pi_U(C)$ and let us consider a language L in which the neighbourhood $I(x) \subseteq U$ is expressible by a formula $patt(x)$, for any $x \in U$. It means that $I(x) = \|patt(x)\|_U \subseteq U$, where $\|patt(x)\|_U$ denotes the meaning of $patt(x)$ restricted to the sample U. In case of rule based classifiers patterns of the form $patt(x)$ are defined by feature value vectors.

We assume that for any new object $x \in U^\infty \setminus U$ we can obtain (e.g., as a result of sensor measurement) a pattern $patt(x) \in L$ with semantics $\|patt(x)\|_{U^\infty} \subseteq U^\infty$. However, the relationships between information granules over U^∞ like sets: $\|patt(x)\|_{U^\infty}$ and $\|patt(y)\|_{U^\infty}$, for different $x, y \in U^\infty$ (or between $\|patt(x)\|_{U^\infty}$ and $y \in U^\infty$), are, in general, known only if they can be expressed by relationships between the restrictions of these sets to the sample U, i.e., between sets $\Pi_U(\|patt(x)\|_{U^\infty})$ and $\Pi_U(\|patt(y)\|_{U^\infty})$.

The set of patterns $\{patt(x) : x \in U\}$ is usually not relevant for approximation of the concept $C \subseteq U^\infty$. Such patterns are too specific or not enough general, and directly can be applied only to a very limited number of new objects. However, by using some generalisation strategies, one can search, in a family of patterns definable from $\{patt(x) : x \in U\}$ in L, for such new patterns that are relevant for approximation of concepts over U^∞. Let us consider a subset $PATTERNS(AS, L, C) \subseteq L$ chosen as a set of pattern candidates for relevant approximation of a given concept C. For example,

[3] For simplicity of reasoning, in this section we use standard definition of approximation spaces (Definition 4).

in case of rule based classifier one can search for such candidate patterns among sets definable by subsequences of feature value vectors corresponding to objects from the sample U. The set $PATTERNS(AS, L, C)$ can be selected by using some quality measures checked on meanings (semantics) of its elements restricted to the sample U (like the number of examples from the concept $\Pi_U(C)$ and its complement that support a given pattern). Then, on the basis of properties of sets definable by these patterns over U we induce approximate values of the inclusion function ν_C on subsets of U^∞ definable by any of such patterns and the concept C.

Next, we induce the value of ν_C on pairs (X, Y) where $X \subseteq U^\infty$ is definable by a pattern from $\{patt(x) : x \in U^\infty\}$ and $Y \subseteq U^\infty$ is definable by a pattern from $PATTERNS(AS, L, C)$.

Finally, for any object $x \in U^\infty \setminus U$ we induce the approximation of the degree $\nu_C(\|patt(x)\|_{U^\infty}, C)$ applying a conflict resolution strategy $Conflict\_res$ (a voting strategy, in case of rule based classifiers) to two families of degrees:

$$\{\nu_C(\|patt(x)\|_{U^\infty}, \|patt\|_{U^\infty}) : patt \in PATTERNS(AS, L, C)\}, \tag{21}$$

$$\{\nu_C(\|patt\|_{U^\infty}, C) : patt \in PATTERNS(AS, L, C)\}. \tag{22}$$

Values of the inclusion function for the remaining subsets of U^∞ can be chosen in any way – they do not have any impact on the approximations of C. Moreover, observe that for the approximation of C we do not need to know the exact values of uncertainty function I_C – it is enough to induce the values of the inclusion function ν_C. Observe that the defined extension ν_C of ν to some subsets of U^∞ makes it possible to define an approximation of the concept C in a new approximation space AS_C by using Definition 5.

In this way, the rough set approach to induction of concept approximations can be explained as a process of inducing a relevant approximation space.

The granule approximation space $\mathcal{AS} = (GS, G, Tr)$ modelling the described process of concept approximations under fixed: U^∞, $C \subseteq U^\infty$, sets of formulas (patterns) $\{patt(x) : x \in U\}$, $PATTERNS(AS, L, C)$, and their semantics $\| \cdot \|_{U^\infty}$ can be defined by

1. GS consisting of the following granules: $C \in P(U^\infty)$, the sample $U \subseteq U^\infty$, $C \cap U$, $U \setminus C$, sets $\|patt(x)\|_U$, defined by $patt(x)$ for any $x \in U$, and the rough inclusion $\nu = \nu_{SRI}$.
2. $G = \{C\}$.
3. The transition relation Tr extending GS to GS' and GS' to GS^*. $Gran(GS')$ is extended from $Gran(GS)$ by the following information granules: sets $\|patt(x)\|_{U^\infty}$, defined by $patt(x)$ for any $x \in U^\infty$, $\|patt\|_{U^\infty}$, for $patt \in PATTERNS(AS, L, C)$. The rough inclusion is extended using estimations described above. GS^* is constructed as follows:
   - $Gran(GS^*) = G \cup$
     $\cup\{(x, \|patt(x)\|_{U^\infty}, \|patt\|_{U^\infty}) : x \in U^\infty \text{and} patt \in PATTERNS(AS, L, C)\}$
   - The rough inclusion ν is extended by:

     $$\nu((x, X, Y), C) = Conflict\_res(\{\nu_C(X, Y) : Y \in \mathcal{Y}\}, \{\nu_C(Y, C) : Y \in \mathcal{Y}\}) \tag{23}$$

     where $X, Y \subseteq U^\infty$, $\mathcal{Y} \subseteq P(U^\infty)$ are sets and the family of sets on which values of ν_C have been estimated in (21) and (22);

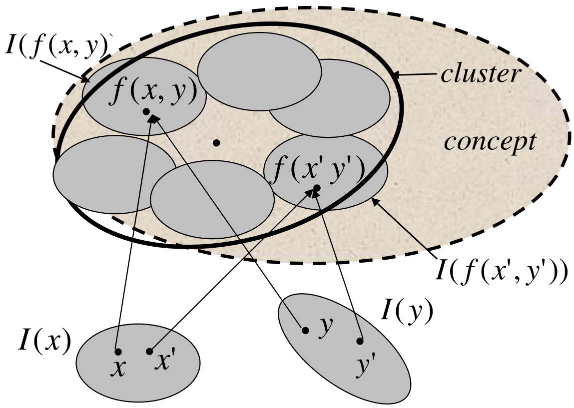

Fig. 2. Relational structure granulation

– The “make granule" operation $\oplus$ satisfies the following constraint:

$$\oplus\{(x,\cdot,\cdot) : x \in Z\} = Z \text{ for any } Z \subseteq C^{\infty}.$$

7 Relational Structure Granulation

In this section we discuss an important role that the relational structure granulation [5], [8] plays in searching for relevant patterns in approximate reasoning, e.g., in searching for relevant approximation patterns (see Section 6 and Figure 2). For any object there is defined a neighbourhood specified by the value of uncertainty function from an approximation space (see Definition 4). From these neighbourhoods some other, more relevant ones (e.g., for the considered concept approximation), should be found. Such neighbourhoods can be extracted by searching the space of neighbourhoods generated from values of uncertainty function by applying some operations to them like generalisation operations, set theoretical operations (union, intersection), clustering and operations on neighbourhoods defined by functions and relations in an underlying relational structure[4]. Figure 2 illustrates an exemplary scheme of searching for neighbourhoods (patterns, clusters) relevant for concept approximation. In the example f denotes a function with two arguments from the underlying relational structure. Due to the uncertainty, we cannot perceive objects exactly but only by using available neighbourhoods defined by the uncertainty function from an approximation space. Hence, instead of the value $f(x,y)$ for a given pair of objects (x,y) one should consider a family of neighbourhoods $\mathcal{F} = \{I(f(x',y')) : (x',y') \in I(x) \times I(y)\}$. From this family $\mathcal{F}$ a subfamily of neighbourhoods, $\mathcal{F}'$, can be chosen that consists of neighbourhoods with some properties relevant for approximation. Next, such subfamily $\mathcal{F}'$ can be generalised to clusters that are relevant for concept approximation, i.e., clusters sufficiently included into the approximated concept (see Figure 2). The inclusion degrees can be measured by granulation of the inclusion function from the relational structure.

[4] Relations from such structure may define relations between objects or their parts.

8 Conclusions

We have discussed problems of approximation space modelling for concept approximation. We also presented consequences for concept approximation of the assumption that information granules involved in concept approximations are perceived by partial information about them. Illustrative examples of approximation spaces were included. We also emphasised the role of relational structure granulation in searching for relevant approximation spaces for concept approximations.

In our further work we would like to use the presented approach for modelling of searching processes for relevant approximation spaces using data and domain knowledge represented, e.g., in natural language.

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