Nearness in Approximation Spaces

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Near To
How near to the bark of a tree are the drifting snowflakes,
swirling gently round, down from winter skies?
How near to the ground are icicles,
slowing forming on window ledges?
...
– Z. Pawlak and J.F. Peters,
Spring, 2002.

Abstract. The problem considered in this paper is the extension of an approximation space to include a nearness relation. Approximation spaces were introduced by Zdzisław Pawlak during the early 1980s as frameworks for classifying objects by means of attributes (features). Pawlak introduced approximations as a means of approximating one set of objects with another set of objects using an indiscernibility relation that is based on a comparison between the feature values of objects. Until now, the focus has been on the overlap between sets. It is possible to introduce a *nearness* relation that can be used to determine the "nearness" of sets that are possibly disjoint. Several members of a family of nearness relations are introduced in this article. The contribution of this article is the introduction of a *nearness* relation that makes it possible to extend Pawlak's model for an approximation space and to consider the extension of generalized approximations spaces.

Keywords: rough sets, approximation, approximation space, classification, feature, nearness relation, neighborhood, proximity space.

1 Introduction

Considerable work on approximation and approximation spaces and their applications has been carried in recent years [7, 8, 11, 14, 17–19, 22–24, 26]. Zdzisław Pawlak introduced approximation spaces during the early 1980s as part of his research on classifying objects by means their feature values [7, 8]. Approximation plays a fundamental role in rough set theory also introduced by Pawlak. There are close ties between approximation and general topology. For example, the approach to *lower approximation* and *upper approximation* introduced by Pawlak is closed related to the topological *interior* and *closure* operators, respectively [14, 20]. Topology is a rich source of constructs that can be used to enrich the original model of an approximation space as well as more recent models of generalized approximation spaces. For example, the *nearness* relation introduced in this article has been inspired by work on proximity spaces [2, 5].

The problem considered in this paper is the extension of the Pawlak model for an approximation space to include a nearness relation [3], [1]. Until now, the focus has been on the overlap between sets. It is possible to introduce a nearness relation that can be used to discover the "nearness" of sets that are possibly disjoint. Several members of a family of nearness relations are introduced in this article. The contribution of this article is the introduction of a nearness relation that makes it possible to extend the Pawlak's model for an approximation space and to consider the extension of generalized approximations spaces.

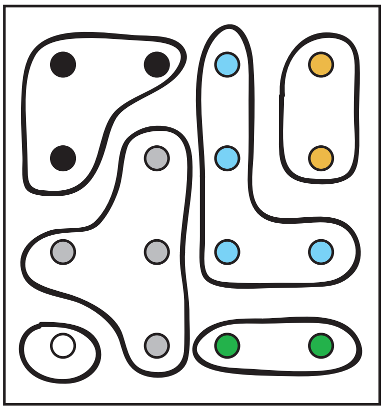
The article is organized as follows. Basic notions and notation for the classification of objects by means of their features that forms the basis for rough set theory are briefly introduced in Sect. 2. Several members of a family of nearness relations are introduced in Sect. 3. Pawlak's model for an approximation space and its extension are briefly presented in Sects. 2 and 4. A nearness form of generalized approximation space is presented in Sect. 4.

2 Rough Sets

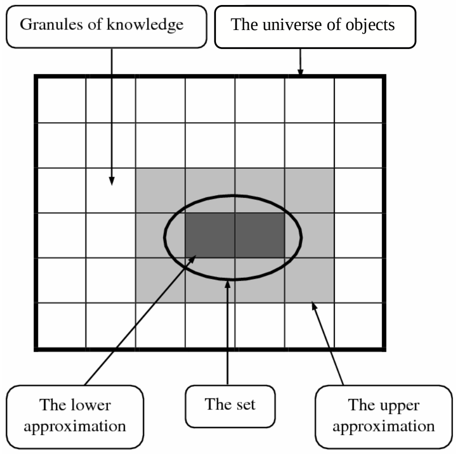
> If we classify objects by means of attributes,
> exact classification is often impossible.
> – Zdzisław Pawlak, January 1981.

> Some categories (subsets of objects) cannot be
> expressed exactly by employing available knowledge.
> Hence, we arrive at the idea of approximation
> of a set by other sets.
> –Zdzisław Pawlak, 1991.

A brief presentation of the foundations of rough set theory is given in this section. Rough set theory has its roots in Zdzisław Pawlak's research on knowledge representation systems during the early 1970s [6]. Rather than attempt to classify objects *exactly* by means of attributes (features), Pawlak considered an approach to solving the object classification problem in a number of novel ways.



1.1: Blocks of Objects



1.2: Sample Set Approximation

Fig. 1. Rudiments of Rough Sets

First, in 1973, he formulated knowledge representation systems (see, e.g., [6]). Then, in 1981, Pawlak introduced approximate descriptions of objects and considered knowledge representation systems in the context of upper and lower classification of objects relative to their attribute values [7]. We start with a system $S = (X, A, V, \sigma)$, where X is a non-empty set of objects, A is a set of attributes, V is a union of sets V_a of values associated with each $a \in A$, and σ is called a knowledge function defined as the mapping $\sigma : X \times A \to V$, where $\sigma(x, a) \in V_a$ for every $x \in X$ and $a \in A$. The function σ is referred to as *knowledge function* about objects from X. The set X is partitioned into elementary sets that later were called blocks, where each elementary set contains those elements of X which have matching attribute values. In effect, a block (elementary set) represents a granule of knowledge (see Fig. 1.2). For example, for any $B \subseteq A$ the B-elementary set for an element $x \in X$ is denoted by $B(x)$, which is defined by

$$B(x) = \{y \in X | \; \forall a \in B \;\; \sigma(x, a) = \sigma(y, a)\} \tag{1}$$

Consider, for example, Fig. 1.1 which represents a system S containing a set X of colored circles and a feature set A that contains only one attribute, namely, *color*. Assume that each circle in X has only one color. Then the set X is partitioned into elementary sets or blocks, where each block contains circles with the same color. In effect, elements of a set $B(x) \subseteq X$ in a system S are classified as *indiscernible* if they are indistinguishable by means of their feature values for any $a \in B$. A set of *indiscernible* elements is called an *elementary set*. Hence, any subset $B \subseteq A$ determines a partition $\{B(x) : x \in X\}$ of X. This partition defines an equivalence relation $Ind(B)$ on X called an *indiscernibility* relation such that $xInd(B)y$ if and only if $y \in B(x)$ for every $x, y \in X$.

Assume that $Y \subseteq X$ and $B \subseteq A$, and consider an approximation of the set Y by means of the attributes in B and B-indiscernible blocks in the partition of X. The union of all blocks that constitute a subset of Y is called the *lower approximation* of Y (usually denoted by B_*Y), representing certain knowledge about Y. The union of all blocks that have non-empty intersection with the set Y is called the *upper approximation* of Y (usually denoted by B^*Y), representing uncertain knowledge about Y. The set $BN_B(Y) = B^*Y - B_*Y$ is called the B-boundary of the set Y. In the case where $BN_B(Y)$ is non-empty, the set Y is a *rough (imprecise)* set. Otherwise, the set Y is a *crisp* set. This approach to classification of objects in a set is represented graphically in Fig. 1.2, where the region bounded by the ellipse represents a set Y, the darkened blocks inside Y represent B_*Y, the gray blocks represent the boundary region $BN_B(Y)$, and the gray and the darkened blocks taken together represent B^*Y.

Consequences of this approach to the classification of objects by means of their feature values have been remarkable and far-reaching. Detailed accounts of the current research in rough set theory and its applications are available, e.g., in [14–16]).

One of the most profound, very important notions underlying rough set theory is approximation. In general, an *approximation* is defined as the replacement of objects by others that resemble the original objects in certain respects [4]. For example, consider a universe U containing objects representing behaviors of agents. In that case, we can consider blocks of behaviors in the partition U/R, where the behaviors within a block resemble (are *indiscernible* from) each other by virtue of their feature values. Then any subset X of U can be approximated by blocks that are either proper subsets of X (lower approximation of the set X denoted R_*X) or by blocks having one or more elements in common with X (upper approximation of the set X denoted R^*X). In rough set theory, the focus is on approximating one set of objects by means of another set of objects based on the feature values of the objects [11]. The lower approximation operator R_* has properties that correspond closely to properties of what is known as the Π_0 topological *interior* operator [8, 20]. Similarly, the upper approximation operator R^* has properties that correspond closely to properties of the Π_0 topological *closure* operator [8, 20]. It was observed in [8] that the key to the rough set approach is provided by the exact mathematical formulation of the concept of approximative (rough) equality of sets in a given approximation space.

3 Nearness Relation in Proximity Spaces Defined by Partitions

This section introduces a proximity space (U, δ) defined relative to a nearness relation δ. It also considers an extension of the approximation space introduced by Zdzisław Pawlak [8] as well as an extension of generalized approximation spaces.

3.1 Basic definition

The basic approach in this section is to define a relation *is near* so that the assertion X *is near* Y is explained relative to features of objects in non-empty subsets $X, Y \subseteq U$, where is U is a non-empty set of objects called a *universe*. In this section, a *nearness* relation δ is defined in a natural way relative to the feature values of objects in U following the convention established by Pawlak for the classification of objects by means of their features. To see this, let $X, Y \subseteq U$ and let $B \subseteq A$ (set of features of objects in U). For simplicity, in what follows, assume that all features of A are real-valued (for symbolic features one can use for example value difference metric [25]). Then define D_B as shown in Eq. 2.

$$D_B(x, y) = \sum_{a \in B} |a(x) - a(y)|. \tag{2}$$

The pseudometric defined by Eqn. (2) can be extended on sets by (3).

$$D_B(X, Y) = inf_{x \in X, y \in Y} D_B(x, y). \tag{3}$$

A requirement of nearness (i.e., only one object $y \in Y$ has feature values that match the feature values of at least one object $x \in X$) and non-nearness can be formulated as shown in (4) and (6), respectively.

$$X \ \delta \ Y \ iff \ \exists y \in Y \ \exists x \in X \ D_B(x, y) = 0. \tag{4}$$

Observe that from this definition we have that for any $X, Y \subseteq U$, it is the case that $X \delta Y$ if and only if there exists $x \in U$ such that $B(x) \cap X \neq \emptyset$ and $B(x) \cap Y \neq \emptyset$.

One can observe that we have: $X \delta Y$ iff $B^*X \cap B^*Y \neq \emptyset$. The upper approximations B^*X, B^*Y are closures of X, Y, respectively, in a topology defined by the B-indiscernibility relation. Hence, (4) defines the standard proximity (see [1]).

Let us recall from [5] a definition of δ-neighborhood in a proximity space (U, δ). We say Y is a proximal or δ-neighborhood of X, written $X \ll Y$, if and only if $X \ \delta \ Y$ and $X \ \delta \ (U - Y)$ is false. We have the following fact characterizing the B-upper approximation of $X \subseteq U$ using the δ-neighborhood relation.

Let (U, δ) be a proximity space where δ is defined by (4). Then for any non-empty subset X of U, we make the assertion in (5).

$$X \ll B^*X, \tag{5}$$

i.e., the upper approximation B^*X of X is a δ-neighborhood of X. The property (5) characterizes the upper approximation of X in terms of proximity space.
In fact, by we have $X \ \delta \ B^*X$ and, by definition of the upper approximation, there is no $y \in U$ such that $B(y) \cap X \neq \emptyset$ and $B(y) \cap (U - B^*X) \neq \emptyset$. Hence, $D_B(X, (U - B^*X)) \neq 0$, i.e., $non(X \ \delta \ (U - B^*X))$.

Let us recall the definition of proximity space.

Definition 1. *Proximity Space [5].*
A binary relation δ on $\mathcal{P}(U)$ (powerset of U) is called a proximity of U iff δ satisfies proximity axioms 1-5. The pair (U, δ) is called a proximity space.

1. *$X\ \delta\ Y$ implies $Y\ \delta\ X$.*
2. *$(X \cup Y)\ \delta\ Z$ implies $X\ \delta\ Z$ or $Y\ \delta\ Z$.*
3. *$X\ \delta\ Y$ implies $X \neq \emptyset$, $Y \neq \emptyset$.*
4. *$X\ \overline{\delta}\ Y$ implies there exists $E \subseteq U$ so that $X\ \overline{\delta}\ E$ and $(U - E)\ \overline{\delta}\ Y$.*
5. *$X\ \cap\ Y \neq \emptyset$ implies $X\ \delta\ Y$.*

Let us observe that (U, δ) is a proximity space for a finite set U, if the relation δ is defined by (4). Under these assumptions it is easy to see that we have $D_B(x, y) = 0$ for some $x \in X$, $y \in Y$ if and only if $D_B(X, Y) = 0$. Hence, we have $X\delta Y$ if and only if $D_B(X, Y) = 0$, where D_B is induced from the pseudometric d_B. In such a case, (U, δ) is a proximity space (see [5], page 8). Assuming that $X\ \overline{\delta}\ Y$ denotes $non(X\ \delta\ Y)$, we have

$$X\ \overline{\delta}\ Y\ iff\ \forall y \in Y\ \forall x \in X\ D_B(x, y) \neq 0. \tag{6}$$

The intuition underlying (4) is that $X\ \delta\ Y$ denotes the fact that the knowledge represented by all of the feature values of one of the objects in Y matches the feature values of one of the objects in the set X. The assertion $X\ \overline{\delta}\ Y$ in (6) says that if there does not exists an object $y \in Y$ whose feature values match the feature values of an object $x \in X$, then X *is not near* Y. No attempt has been made to give a measure of the *degree of nearness*, which is outside the scope of this article.

3.2 Relations between nearness and approximations

> The key to the presented approach
> is provided by the exact mathematical formulation,
> of the concept of approximative (rough) equality
> of sets in a given approximation space.
> –Zdzisław Pawlak, 1982.

In this section, we briefly consider an extension the approximation space introduced in [8] with the nearness relation δ. In [11], an approximation space is represented by the pair (U, R), where U is a universe of objects, and $R \subseteq U \times U$ is an indiscernibility relation (denoted Ind as in Sect. 2) defined by an attribute set (i.e., $R = Ind(A)$ for some attribute set A). In this case, R is an equivalence relation. Let $[x]_R$ denote an equivalence class of an element $x \in U$ under the indiscernibility relation R, where $[x]_R = \{y \in U : xRy\}$.

In this context, R-approximations of any set $X \subseteq U$ are based on the exact (crisp) containment of sets. Then set approximations are defined as follows:

- $x \in U$ belongs with certainty to $X \subseteq U$ (i.e., x belongs to the R-lower approximation of X), if $[x]_R \subseteq X$.

- $x \in U$ possibly belongs $X \subseteq U$ (i.e., x belongs to the R-upper approximation of X), if $[x]_R \cap X \neq \oslash$.
- $x \in U$ belongs with certainty neither to the X nor to $U - X$ (i.e., x belongs to the R-boundary region of X), if $[x]_R \cap (U - X) \neq \oslash$ and $[x]_R \cap X \neq \oslash$.

A *nearness approximation space* is represented by (U, R, δ), where δ is the nearness relation defined in (4) in Sect. 3 and R is the indiscernibility relation defined by a set of attributes B. What follows is a selection of properties of this form of an approximation space.

1. $B(x)\ \delta\ X$ for any $B(x) \subseteq B_* X$, since $B_*(x) \cap X \neq \emptyset$.
2. $B(x)\ \delta\ B^* X$ for any $B(x) \subseteq B^* X$. Similarly, we have
3. $B(x)\ \delta\ B_* X$ for any $B(x) \subseteq B_* X$.
4. $B_*(x)\ \delta\ B^* X$, since $B_*(x)\ \subseteq\ B^* X$.
5. $BN_B(X)\ \overline{\delta}\ B_* X$, since $BN_B(X)\ \cap\ B_* X\ =\ \emptyset$, and there is no object in $B_* X$ that matches an object in $BN_B(X)$.

3.3 Some remarks on non-standard proximities

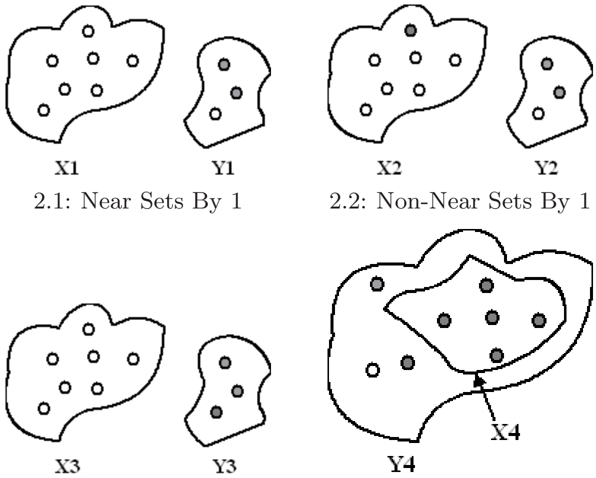
Observe, that one can define a less restrictive concept of nearness. That is, the assertions "*X is near Y*" and "*X is not near Y*" can be defined as shown in (7) and (8), respectively.

$$X\ \delta\ Y\ iff\ \exists y \in Y\ \forall x \in X,\ D_B(x, y) = 0. \tag{7}$$

$$X\ \overline{\delta}\ Y\ iff\ \forall y \in Y\ \exists x \in X\ D_B(x, y) \neq 0. \tag{8}$$

The formulation of *nearness* in (7) asserts that if there exists an object y in set Y whose feature values match the feature values of every object x in set X, then X *is near* Y. The intuition underlying (7) is that $X\ \delta\ Y$ (i.e., X *is near* Y) whenever the knowledge represented by all of the feature values of one of the objects in Y matches the feature values of all of the objects in the set X (for simplicity we will write δ ($\overline{\delta}$) instead of δ_B ($\overline{\delta_B}$)). In defining nearness of one set of objects to another set of objects, definition (7) prescribes a minimum condition for *nearness*, namely, only one of the objects in one set must have feature values that match those of all of the objects in the other set. The assertion $X\ \overline{\delta}\ Y$ in (8) says that if there does not exists an object $y \in Y$ whose feature values match the feature values of every object $x \in X$, then X *is not near* Y. In effect, $X\ \overline{\delta}\ Y$ occurs in the case where it is determined that no object in Y matches our knowledge of all of the objects in X.

Example 1. **Sets of Colored Circles**.
By way of illustration of the *nearness* relation, consider δ as defined in (7). Let X, Y denote two sets of circles and assume $B \subseteq A$ contains one feature, namely,



2.1: Near Sets By 1 2.2: Non-Near Sets By 1

2.3: Other Non-Near Sets 2.4: Near Subset

Fig. 2. Sample Sets of Objects

brightness (i.e., descriptor of visual perception). Further, let the brightness of each circle be represented by a number indicating the average brightness of the pixels contained in a circle in a greyscale image (0 for white, and 1 for black). Examples are given in Fig. 2. In Fig. 2, assume that each circle is made up of all white or all black pixels. Then, for example in Fig. 2.1, $X1\ \delta\ Y1$ (i.e., $X1$ is near $Y1$ because there is a white circle in Y with an average brightness that matches the average brightness of all of the circles in X). That is, $D(X1, Y1) = 0$. Similarly, in Fig. 2.4, $X4\ \delta\ Y4$, since $D(X4, Y4) = 0$. This happens because either one of the black circles in $Y4$ has an average brightness matches the average brightness of all of the circles in $X4$. In Fig. 2.2, $X2\ \overline{\delta}\ X2$, since $D(X2, Y2) \neq 0$. That is, there is no circle in $Y2$ with an average brightness that matches the average brightness of all of the circles in $X2$. The presence of mixed circles in both sets prevents *nearness*. Again, for example, $X3$ and $Y3$ in Fig. 2.3 have no matching circles (i.e., the circles in the two sets have non-matching colors). Hence, $D(X3, Y3) \neq 0$, i.e., $X3\ \overline{\delta}\ Y3$. Notice that if δ is defined as in (4), then $X2\ \delta\ X2$.

Let us observe that δ defined in (7), in general, fails to satisfy axiom 1 of a proximity space (see Def. 1). To see this, consider, for example, $X1\ \delta\ Y1$, but $Y1\ \overline{\delta}\ X1$ (i.e. $Y1$ is not near $X1$) in Fig. 2.1, since there is no object in $X1$ that matches all of the objects in $Y1$.

4 Nearness in Generalized Approximation Spaces

Several generalizations of the classical rough set approach based on approximation spaces defined as pairs of the form (U, R), where R is the equivalence relation (called an indiscernibility relation) on the non-empty set U, have been reported in the literature (see, e.g., [22, 24, 26, 14–16]). Let us mention two of them.

A generalized approximation space can be defined by a tuple $GAS = (U, N, \nu)$ where N is a *neighborhood function* defined on U with values in the powerset $\mathcal{P}(U)$ of U (i.e., $N(x)$ is the *neighborhood* of x) and ν is the *overlap function* defined on the Cartesian product $\mathcal{P}(U) \times \mathcal{P}(U)$ with values in the interval $[0, 1]$ measuring the degree of overlap of sets. The lower GAS_* and upper GAS^* approximation operations can be defined in a GAS by Eqs. 9 and 10.

$$GAS_*(X) = \{x \in U : \nu(N(x), X) = 1\}, \tag{9}$$
$$GAS^*(X) = \{x \in U : \nu(N(x), X) > 0\}. \tag{10}$$

In the standard case, $N(x)$ equals the equivalence class $B(x)$ or block of the indiscernibility relation $Ind(B)$ for a set of features B. In the case where R is a tolerance (similarity) relation[4], $\tau \subseteq U \times U$, we take $N(x) = \{y \in U : x\tau y\}$, i.e., $N(x)$ equals the tolerance class of τ defined by x. The standard inclusion relation ν_{SRI} is defined for $X, Y \subseteq U$ by Eq. 11.

$$\nu_{SRI}(X, Y) = \begin{cases} \frac{|X \cap Y|}{|Y|}, & \text{if Y} \neq \emptyset, \\ 1, & \text{otherwise.} \end{cases} \tag{11}$$

For applications, it is important to have some constructive definitions of N and ν.

One can consider another way to define $N(x)$. Usually together with a GAS, we consider some set F of formulas describing sets of objects in the universe U of the GAS defined by semantics $\|\cdot\|_{GAS}$, i.e., $\|\alpha\|_{GAS} \subseteq U$ for any $\alpha \in F$. Now, one can take the set the neighborhood function as shown in Eq. 12.

$$N_F(x) = \{\alpha \in F : x \in \|\alpha\|_{GAS}\}, \tag{12}$$

and $N(x) = \{\|\alpha\|_{GAS} : \alpha \in N_F(x)\}$. Hence, more general uncertainty functions having values in $\mathcal{P}(U)$ can be defined and as a consequence different definitions of approximations are considered. For example, one can consider the following definitions of approximation operations in GAS defined in Eqs. 13 and 14.

$$GAS_\circ(X) = \{x \in U : \nu(Y, X) = 1 \text{ for some } Y \in N(x)\}, \tag{13}$$
$$GAS^\circ(X) = \{x \in U : \nu(Y, X) > 0 \text{ for any } Y \in N(x)\}. \tag{14}$$

There are also different forms of rough inclusion functions. Let us consider two examples.

[4] Recall that a *tolerance* is a binary relation $R \subseteq U \times U$ on a set U having the reflexivity and symmetry properties, i.e., xRx for all $x \in U$ and xRy implies yRx for all $x, y \in U$.

In the first example of a rough inclusion function, a threshold $t \in (0, 0.5)$ is used to relax the degree of inclusion of sets. The rough inclusion function ν_t is defined by Eq. 15.

$$\nu_t(X,Y) = \begin{cases} 1, & \text{if} \quad \nu_{SRI}(X,Y) \geq 1-t, \\ \frac{\nu_{SRI}(X,Y)-t}{1-2t}, & \text{if} \quad t \leq \nu_{SRI}(X,Y) < 1-t, \\ 0, & \text{if} \quad \nu_{SRI}(X,Y) \leq t. \end{cases} \tag{15}$$

One can obtain approximations considered in the variable precision rough set approach (VPRSM) by substituting in (9)-(10) the rough inclusion function ν_t defined by (15) instead of ν, assuming that Y is a decision class and $N(x) = B(x)$ for any object x, where B is a given set of attributes.

Another example of application of the standard inclusion was developed by using probabilistic decision functions.

The rough inclusion relation can be also used for function approximation and relation approximation. In the case of function approximation the inclusion function ν^* for subsets $X, Y \subseteq U \times U$, where $X, Y \subseteq \mathcal{R}$ and $\mathcal{R}$ is the set of reals, is defined by Eq. 16.

$$\nu^*(X,Y) = \begin{cases} \frac{card(\pi_1(X \cap Y))}{card(\pi_1(X))}, & \text{if } \pi_1(X) \neq \emptyset, \\ 1, & \text{if } \pi_1(X) = \emptyset, \end{cases} \tag{16}$$

where π_1 is the projection operation on the first coordinate. Assume now, that X is a cube and Y is the graph $G(f)$ of the function $f : \mathcal{R} \longrightarrow \mathcal{R}$. Then, e.g., X is in the lower approximation of f if the projection on the first coordinate of the intersection $X \cap G(f)$ is equal to the projection of X on the first coordinate. This means that the part of the graph $G(f)$ is "well" included in the box X, i.e., for all arguments that belong to the box projection on the first coordinate the value of f is included in the box X projection on the second coordinate.

A *generalized nearness approximation space* is represented by $NGAS = (U, N, \nu, \delta)$, where δ is the nearness relation defined by (17).

$$X \ \delta \ Y \leftrightarrow \exists x \in X \ \exists y \in Y \ (y \in N(x)). \tag{17}$$

Observe that, in general, the proximity relation defined by (17) does not satisfy the axiom (1) of a proximity space (see Def. 1). However, assuming that $x \in N(x)$ and $y \in N(x)$ iff $x \in N(y)$ for any $x, y \in U$, and additionally $X \delta Y$ for $X = Y = \emptyset$ we obtain that the relation $\delta \subseteq P(U) \times P(U)$ defined by (17) is a tolerance relation [3]. For example, for a given $\varepsilon \in [0, 1]$, a set of attributes B, and $x \in U$ one can define the neighborhood of x by $N_B^{\varepsilon}(x) = \{y \in U : \sum_{a \in B} \frac{|a(x)-a(y)|}{max(a)-min(a)} \leq \varepsilon\}$.

What follows is a selection of properties of this form of an approximation space.

1. $GAS_*(X) \ \delta \ \{x\}$ for every $x \in GAS_*(X)$.
2. $GAS_*(X) \ \delta \ N(x)$, since $x \in N(x)$ will also be an object in $GAS_*(X)$.

3. $GAS_*(X)\ \delta\ GAS^*(X)$ if $GAS_*(X) \neq \emptyset$.
4. $N(x)\ \delta\ \{x\}$ for $x \in\ GAS_\circ(X)$.
5. $GAS_\circ(X)\ \delta\ GAS^\circ(X)$ if $GAS_\circ(X) \neq \emptyset$.

Usually families of approximation spaces labelled by some parameters are considered. By tuning such parameters according to chosen criteria (e.g., minimal description length), one can search for the optimal approximation space for a concept description.

Conclusion

This paper has presented an extension of approximation spaces introduced by Pawlak as well as generalized approximation spaces based on the introduction of a nearness relation. The significance of these extensions is that we now have the possibility of measuring our knowledge about objects based on a perception of the "nearness" of objects classified by means of their feature values. The proposed extension is a direct outgrowth of the basic approach to classifying objects proposed by Zdzisław Pawlak during the early 1980s. The negation of the nearness relation (i.e., $\overline{\delta}$) suggests the possibility of extending the model for conflict also proposed by Pawlak, which is based on the negation of the indiscernibility relation (see, e.g., [10, 12, 13]). The aim of the paper is also to look for proximity to a degree.

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