# Satisfiability and Meaning in Approximation Spaces

Anna Gomolińska*

University of Białystok, Department of Mathematics,
Akademicka 2, 15267 Białystok, Poland,
anna.gom@math.uwb.edu.pl

**Abstract.** In this paper, we study general notions of satisfiability and meaning of formulas and sets of formulas in approximation spaces. We also touch upon derivative concepts of meaning and applicability of rules. Approximate satisfiability and meaning are important, among others, for modeling of complex systems like systems of adaptive social agents. Rather than proposing one particular form of rough satisfiability and meaning, we present a number of alternative approaches.

## 1 Introduction

In this article, we focus upon satisfiability and meaning of formulas and sets of formulas for objects of an approximation space (AS), where an AS is understood in the sense of Skowron and Stepaniuk [20, 21]. We view satisfiability and meaning of formulas/sets of formulas as two faces of the same coin. They are interdefinable: A formula is satisfied for an object iff the object belongs to the meaning of the formula, and analogously for a set of formulas. Although satisfiability and meaning of formulas are logical notions, they have interesting practical implications. Serving as building blocks in defining of the meaning of rules and sets of rules, they contribute to such practical issues as application and quality of rules and their sets. Apart from satisfiability and meaning, we briefly address the derivative problem of applicability of rules as well.

ASs are particularly rich and fruitful as a field of exploration and exploitation of various approximate notions of satisfiability and meaning. Defining and, next, studying these notions is not merely a matter of art. We can use them to model reasoning modules of agents interacting in an open environment under risk, uncertainty, incomplete knowledge, etc. In this paper, we study satisfiability and meaning in general terms, aiming at presentation of the general perspective, mechanisms, and richness of possible definitions of "rough" satisfiability and meaning. We recall some particular cases, already described in the literature,

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and we introduce some new ones deserving an attention in our opinion. Which notion of satisfiability, meaning, or applicability to choose is not only a matter of taste but, first of all, it should depend on the intended application area.

Section 2 serves as an introduction to the topics of ASs. In Sect. 3, we analyze the problem of satisfiability and meaning of formulas in ASs. Section 4 is devoted to satisfiability and meaning of sets of formulas. In Sect. 5, general notions of meaning and applicability of rules are proposed. The last section contains concluding remarks.

## 2 ASs and Information Granules

Historically, the notion of an AS in the rough sense may be traced back to the Pawlak *information systems* (ISs for short) [10, 11, 13]. An IS is a pair $\mathcal{A} = (U, A)$ of finite non-empty sets of objects and attributes, respectively. Objects and attributes are denoted by $u$ and $a$, respectively, with subscripts if needed. Every attribute $a$ is a mapping on $U$ assigning to each object $u$ of $\mathcal{A}$, a value $a(u) \in V_a$. Values of attributes, i.e., elements of $W = \bigcup\{V_a \mid a \in A\}$ are denoted by $v$, possibly with subscripts. Henceforth, by $\wp U$ and $(\wp U)^2$ we shall denote the power set of $U$ and the Cartesian product $\wp U \times \wp U$, respectively. With each set $B \subseteq A$, there is associated an equivalence relation $\mathrm{ind}_B \subseteq U^2$ of $B$-indiscernibility of objects such that for any $u, u' \in U$, $(u, u') \in \mathrm{ind}_B$ iff $\forall a \in B. a(u) = a(u')$. Relation $\mathrm{ind}_B$ induces a mapping $\Gamma_B : U \mapsto \wp U$ such that for any $u \in U$, $\Gamma_B u = \{u' \in U \mid (u', u) \in \mathrm{ind}_B\}$. Clearly, the image of $U$ given by $\Gamma_B$, $\Gamma_B^{\rightarrow} U$, is a partition of $U$. In this way, $\mathcal{A}$ gives rise to the mutually definable structures $(U, \mathrm{ind}_B)$ and $(U, \Gamma_B)$. Within these structures, concepts (i.e., sets of objects) of $U$ may be approximated by means of the *lower* and *upper rough $B$-approximation* mappings $\mathrm{low}^{\cup}_B, \mathrm{upp}^{\cup}_B : \wp U \mapsto \wp U$, respectively, where for any set of objects $x$,

$$\mathrm{low}^{\cup}_B x = \bigcup\{\Gamma_B u \mid \Gamma_B u \subseteq x\} \text{ and } \mathrm{upp}^{\cup}_B x = \bigcup\{\Gamma_B u \mid \Gamma_B u \cap x \neq \emptyset\}. \quad (1)$$

Here, $\mathrm{low}^{\cup}_B x$ and $\mathrm{upp}^{\cup}_B x$ are the *lower* and *upper rough $B$-approximations* of $x$, respectively. Thus, $(U, \Gamma_B)$ (or, equivalently, $(U, \mathrm{ind}_B)$) may be called a *rough approximation space*, where elementary building blocks, used to approximate concepts, are sets $\Gamma_B u$. A natural, immediate generalization consists in taking any equivalence relation $\varrho \subseteq U^2$ instead of $\mathrm{ind}_B$. Then, $\Gamma_\varrho$ is defined by $\Gamma_\varrho u \stackrel{\mathrm{def}}{=} \varrho^{\leftarrow}\{u\}$ for any $u \in U$. To the same effect, one may start with a mapping $\Gamma : U \mapsto \wp U$ such that for any $u \in U$, $u \in \Gamma u$ and $\Gamma^{\rightarrow} U$ is a partition of $U$. Lower and upper rough approximation mappings $\mathrm{low}^{\cup}, \mathrm{upp}^{\cup}$ may be defined as in (1).

The next step towards generalization of the notion of AS turned out to be crucial from the perspective of further developments and applications. In 1994, Skowron and Stepaniuk introduced the notion of a *parameterized approximation space* [20, 21]. In a recent formulation, any such space is a triple of the form $\mathcal{M}_\$ = (U, \Gamma_\$, \kappa_\$)$, where $U$ is a non-empty set of objects, $\Gamma_\$ : U \mapsto \wp U$ is an *uncertainty* mapping, and $\kappa_\$ : (\wp U)^2 \mapsto [0, 1]$ is a rough inclusion function

(RIF), equipped with a list of tuning parameters \$ to obtain a satisfactory quality of approximation. We omit the parameters for simplicity. Elements of $U$, called objects and denoted by $u$ with subscripts if needed, are known by their properties only. Objects similar to an object $u$ form a set being a *granule of information* in the sense of Zadeh [25]. Indiscernibility may be viewed as a special case of similarity. In particular, every object is similar to itself. The view of the universe $U$ as being covered by elementary granules of information is just realized by the uncertainty mapping $\Gamma$ which assigns to every object $u$, a set $\Gamma u$ of objects similar to $u$, called an *elementary* granule of information. By assumption, $u \in \Gamma u$. A set of objects is *definable* if it is a union of elementary granules. In general, a granule of information may or may not be a subset of the universe $U$ of $\mathcal{M}$. Just as well, it may be a possibly ordered family of sets of objects or a hierarchical structure built upon $U$ as, e.g., a complex of objects of $\mathcal{M}$ [5]. In our terminology, any mapping which assigns to objects or to set-theoretical constructs obtained from objects (e.g., sets of objects or families of sets of objects), some granules of information is called a *granulation mapping*. Thus, an uncertainty mapping is an example of a granulation mapping. Information granules form, possibly hierarchical, systems of granules. Discovery of information granules satisfying a given specification, computation on granules, degrees of closeness and inclusion of granules are examples of issues of interest [17, 18, 22, 23].

Broadly speaking, RIFs are mappings which measure degrees of containment of sets of objects in sets of objects. The formal notion of a RIF is characterized by the axioms of *rough mereology*, a theory of the notion of being-a-part-in-degree, proposed by Polkowski and Skowron [15, 16]. They generalized Leśniewski's mereology [8], a formal theory of the relationship of being-a-part. The most famous RIFs are the *standard* ones, based on a frequency count in line with Łukasiewicz's idea [9]. Where $U$ is finite, the standard RIF is a mapping $\kappa^{\pounds} : (\wp U)^2 \mapsto [0,1]$ such that for any $x, y \subseteq U$, $\kappa^{\pounds}(x,y) = \begin{cases} \frac{\#(x \cap y)}{\#x} & \text{if } x \neq \emptyset \\ 1 & \text{otherwise.} \end{cases}$ In the case of possibly infinite $U$, by a *quasi-standard* RIF we mean any RIF which for finite first arguments is defined as above. In particular, $\kappa^{\pounds}$ is quasi-standard. In our approach, a RIF over $U$ is a mapping $\kappa : (\wp U)^2 \mapsto [0,1]$ satisfying (A1)–(A3) for any $x, y, z \subseteq U$: (A1) $\kappa(x,y) = 1$ iff $x \subseteq y$; (A2) If $x \neq \emptyset$, then $\kappa(x,y) = 0$ iff $x \cap y = \emptyset$; (A3) If $y \subseteq z$, then $\kappa(x,y) \leq \kappa(x,z)$. (A1), (A3) are in line with axioms of rough mereology, whereas (A2) is stronger. As expected, $\kappa^{\pounds}$ satisfies the postulates.

In $\mathcal{M} = (U, \Gamma, \kappa)$, sets of objects (concepts) may be approximated in varied ways (see, e.g., [4] for a discussion and references to the literature). According to the Skowron – Stepaniuk approach, every concept $x \subseteq U$ may be approximated by means of the *lower* and *upper rough approximation mappings* $\mathrm{low}, \mathrm{upp} : \wp U \mapsto \wp U$, respectively, such that

$$\mathrm{low}x = \{u \in U \mid \kappa(\Gamma u, x) = 1\} \text{ and } \mathrm{upp}x = \{u \in U \mid \kappa(\Gamma u, x) > 0\}. \tag{2}$$

Like earlier, $\mathrm{low}x$ and $\mathrm{upp}x$ are called the *lower* and *upper rough approximations* of $x$, respectively. By (A1)–(A3), it holds that $\mathrm{low}x = \{u \in U \mid \Gamma u \subseteq x\}$ and

$\text{upp}x = \{u \in U \mid \Gamma u \cap x \neq \emptyset\}$. Moreover, $\text{low}^{\cup} x = \bigcup \Gamma^{\rightarrow}\text{low}x$ and $\text{upp}^{\cup} x = \bigcup \Gamma^{\rightarrow}\text{upp}x$.

Ziarko [26, 27] refined the Pawlak rough set model by introducing degrees of precision of approximation. In his model, known as the *variable-precision rough set model*, concepts are approximated in terms of $t$-positive and $t$-negative regions of sets of objects, where $t \in [0, 1]$. In line with (2), the mappings of *$t$-positive* and *$t$-negative regions* of sets of objects, $\text{pos}_t, \text{neg}_t : \wp U \mapsto \wp U$, respectively, may be defined as follows, for any set of objects $x$:

$$\text{pos}_t x \stackrel{\text{def}}{=} \{u \in U \mid \kappa(\Gamma u, x) \geq t\} \text{ and } \text{neg}_t x \stackrel{\text{def}}{=} \{u \in U \mid \kappa(\Gamma u, x) \leq t\}. \quad (3)$$

$\text{pos}_t x, \text{neg}_t x$ are the *$t$-positive* and *$t$-negative regions* of $x$, respectively. Like in the case of lower and upper approximation mappings, the mappings of $t$-positive and $t$-negative regions of sets of objects may also be understood as mappings $\text{pos}^{\cup}_t, \text{neg}^{\cup}_t : \wp U \mapsto \wp U$, respectively, such that for any set of objects $x$,

$$\begin{aligned} \text{pos}^{\cup}_t x &= \bigcup \Gamma^{\rightarrow}\text{pos}_t x = \bigcup \{\Gamma u \mid \kappa(\Gamma u, x) \geq t\}; \\ \text{neg}^{\cup}_t x &= \bigcup \Gamma^{\rightarrow}\text{neg}_t x = \bigcup \{\Gamma u \mid \kappa(\Gamma u, x) \leq t\}. \end{aligned} \quad (4)$$

$\text{pos}^{\cup}_t x, \text{neg}^{\cup}_t x$ are just the *$t$-positive* and *$t$-negative regions* of $x$ in the sense of Ziarko, respectively.[1] Notice that $\text{low}x = \text{pos}_1 x$, $\text{low}^{\cup} x = \text{pos}^{\cup}_1 x$, and $\text{upp}x = U - \text{neg}_0 x$.

## 3 Satisfiability and Meaning of Formulas

Assume that a formal language L is given to express properties of $\mathcal{M}$. Formulas of L are denoted by the lowercase Greek letters $\alpha, \beta, \gamma$, possibly with subscripts. The set of all formulas of L is denoted by FOR. In general, a relation of *satisfiability* of formulas of L for objects of $\mathcal{M}$ is a relation $\sigma \subseteq U \times \text{FOR}$, where for any object $u$ and a formula $\alpha$, $(u, \alpha) \in \sigma$ reads as "$\alpha$ is *satisfied* for $u$ in the sense of $\sigma$".[2] $\sigma$ induces mappings $||\cdot||_\sigma : \text{FOR} \mapsto \wp U$ and $|\cdot|_\sigma : U \mapsto \wp\text{FOR}$ such that for any $u \in U$ and $\alpha \in \text{FOR}$,

$$\begin{aligned} ||\alpha||_\sigma &= \{u \in U \mid (u, \alpha) \in \sigma\} = \sigma^{\leftarrow}\{\alpha\}; \\ |u|_\sigma &= \{\alpha \in \text{FOR} \mid (u, \alpha) \in \sigma\} = \sigma^{\rightarrow}\{u\}. \end{aligned} \quad (5)$$

The set of all objects for which $\alpha$ is satisfied in the sense of $\sigma$, i.e., $||\alpha||_\sigma$, is called the *meaning (extension)* of $\alpha$ in the sense of $\sigma$. $|u|_\sigma$ consists of all formulas satisfied for $u$ in the sense of $\sigma$. Both $||\alpha||_\sigma$ and $|u|_\sigma$ are examples of granules of information formed on the base of functionality.

On the other hand, we may start with a notion of meaning of a formula. Suppose a mapping $||\cdot|| : \text{FOR} \mapsto \wp U$ is given which assigns to every formula

[1] See, e.g., [27] for the probability-based variable-precision rough set model, not considered here.

[2] Equivalently, $\alpha$ is *true* of (*holds* for) $u$ in the sense of $\sigma$.

$\alpha$, a set of objects for which $\alpha$ is satisfied. $||\alpha||$ is referred to as the *meaning (extension)* of $\alpha$ in the sense of $||\cdot||$. The mapping $||\cdot||$ induces a satisfiability relation $\sigma_{||\cdot||} \subseteq U \times \mathrm{FOR}$ such that for every $u$ and $\alpha$ as earlier,

$$(u, \alpha) \in \sigma_{||\cdot||} \text{ iff } u \in ||\alpha||. \tag{6}$$

Notice that $||\cdot||_{\sigma_{||\cdot||}} = ||\cdot||$ and $\sigma_{||\cdot||_\sigma} = \sigma$. We can also say that $\alpha$ is *satisfiable* in a given sense in $\mathcal{M}$ iff its meaning in the very sense is non-empty; and $\alpha$ is *unsatisfiable* in $\mathcal{M}$ otherwise.

The general notions of satisfiability and meaning of formulas in ASs comprise a great variety of particular cases. To give concrete examples, let us start with an IS $\mathcal{A} = (U, A)$ and define an uncertainty mapping $\Gamma : U \mapsto \wp U$ determining elementary granules of information associated with objects of $U$. Given a RIF $\kappa : (\wp U)^2 \mapsto [0, 1]$, we finally arrive at an AS $\mathcal{M} = (U, \Gamma, \kappa)$. A propositional language L may be defined as in [14, 17]. For simplicity, objects and attributes are identified with their names and the only terms are the constant symbols being elements of $A \cup W$. The auxiliary symbols are the parentheses (, ) and commas. Atomic formulas are pairs of terms of the form $(a, v)$, called *descriptors*, where $a \in A$ and $v \in W$. Primitive connectives are $\wedge$ (conjunction) and $\neg$ (negation). The remaining propositional connectives like $\vee$ (disjunction), $\rightarrow$ (material implication), and $\leftrightarrow$ (double implication) are classically defined by means of $\wedge, \neg$. Compound formulas are formed from the atomic ones as usual. A relation of crisp satisfiability of formulas for objects, $\models$, may be defined as follows, for any formulas $(a, v), \alpha, \beta$ and an object $u$:

$$\begin{aligned} &u \models (a, v) \text{ iff } a(u) = v. \\ &u \models \alpha \wedge \beta \text{ iff } u \models \alpha \text{ and } u \models \beta. \\ &u \models \neg\alpha \text{ iff } u \not\models \alpha. \end{aligned} \tag{7}$$

The corresponding mapping of meaning is denoted by $||\cdot||$ for simplicity. Thus, $||(a, v)|| = \{u \in U \mid a(u) = v\}$, $||\alpha \wedge \beta|| = ||\alpha|| \cap ||\beta||$, and $||\neg\alpha|| = U - ||\alpha||$.

In [7], rough graded forms of the crisp notions of satisfiability and meaning were studied. Let us briefly recall the main idea. For any $t \in [0, 1]$, a formula $\alpha$ is *t-satisfied* for $u$,[3] $u \models_t \alpha$, iff $\kappa(\Gamma u, ||\alpha||) \geq t$.[4] In the standard case, it means that $\alpha$ is satisfied for sufficiently many objects similar to $u$. Hence, the corresponding *t-meaning* of $\alpha$ is the set

$$||\alpha||_t = \{u \in U \mid u \models_t \alpha\}. \tag{8}$$

Thus, whether or not $\alpha$ is $t$-satisfied for $u$ entirely depends on the crisp satisfiability of $\alpha$ for objects similar to $u$. Such a form of graded satisfiability, called *fundamental*, may be useful if we cannot determine whether or not a particular

[3] Equivalently, $\alpha$ is *true* of $u$ or *holds* for $u$ in degree $t$.

[4] Due to the very natural form of $t$-satisfiability it is difficult to say who mentioned this notion first. However, its generalization to the case of a set of formulas was proposed by the author of this article in [7].

formula holds for an object $u$ but we know that it is satisfied for sufficiently many objects similar to $u$. Observe that the mapping of $t$-meaning $||\cdot||_t$ is the composition of the mapping of $t$-positive region $\mathrm{pos}_t$ and $||\cdot||$, viz., $||\cdot||_t = \mathrm{pos}_t \circ ||\cdot||$. Thus, the $t$-meaning of a formula is the $t$-positive region of its crisp meaning.

The fundamental notion of graded satisfiability of $\alpha$ for $u$ may be enhanced by requiring that $\alpha$ be satisfied for $u$ both in the crisp sense and in degree $t \in [0, 1]$. In this case, the meaning of $\alpha$ is defined as the set $||\alpha||_{c+t} = ||\alpha|| \cap ||\alpha||_t$.

To formalize the idea that a formula $\alpha$ rather holds (than not) in a degree for a given $u$, another refinement of the fundamental $t$-satisfiability can prove useful if the underlying logic is non-classical or the RIF is not standard. For any $\bar{t} = (t_1, t_2)$, where $0 \leq t_1 < t_2 \leq 1$, let

$$u \models^{np}_{\bar{t}} \alpha \stackrel{\mathrm{def}}{\leftrightarrow} \kappa(\mathit{\Gamma} u, ||\neg\alpha||) \leq t_1 \text{ and } \kappa(\mathit{\Gamma} u, ||\alpha||) \geq t_2.$$

$$\text{Hence, } ||\alpha||^{np}_{\bar{t}} \stackrel{\mathrm{def}}{=} \mathrm{neg}_{t_1}||\neg\alpha|| \cap \mathrm{pos}_{t_2}||\alpha||. \tag{9}$$

Informally speaking, a formula is *$\bar{t}$-satisfied* for $u$, $u \models^{np}_{\bar{t}} \alpha$, iff a sufficiently large part of the set $\mathit{\Gamma} u$ of objects similar to $u$ "votes" against $\neg\alpha$ and a sufficiently large part $\mathit{\Gamma} u$ "votes" for $\alpha$, where sufficiency is determined by $\bar{t}$. If the underlying logic is classical and $\kappa$ is standard, then the above notions resolve themselves into the fundamental one. In this case, $\kappa(\mathit{\Gamma} u, ||\alpha||) + \kappa(\mathit{\Gamma} u, ||\neg\alpha||) = 1$ and, in the sequel, $\mathrm{neg}_{t_1}||\neg\alpha|| = \mathrm{pos}_{1-t_1}||\alpha||$.

In [12], Pawlak introduced a notion of *rough truth* of formulas in a rough AS with a finite universe $U$, partitioned by an equivalence relation $\varrho$.[5] This notion has been further developed and studied by Chakraborty and Banerjee [1, 2]. In our formulation, $\alpha$ is *roughly true* in $\mathcal{M}$ iff $\mathrm{upp}^{\cup}||\alpha|| = U$.[6] Although the underlying notions of rough satisfiability and meaning of formulas have not been defined explicitly, one can say that $\alpha$ is *roughly satisfied* for $u$ iff $u \in \mathrm{upp}^{\cup}||\alpha||$. The correponding meaning of $\alpha$ is the set $||\alpha||^P = \mathrm{upp}^{\cup}||\alpha||$.

Yet another concept of graded satisfiability of formulas in ASs has been proposed by Wolski (the author of an interesting paper on the relationships among major theories of qualitative data analysis and Galois connections [24]) in an unpublished manuscript. For any $t \in [0, 1]$, the relation of *$t$-satisfiability* in the sense of Wolski, $\models^W_t$, is defined as follows:

$$u \models^W_t \alpha \text{ iff } \forall u' \in U.\forall t' \in [t, 1].(\kappa(\mathit{\Gamma} u', \mathit{\Gamma} u) \geq t' \Rightarrow u' \models \alpha).$$

$$\text{Hence, } ||\alpha||^W_t = \{u \in U \mid u \models^W_t \alpha\}. \tag{10}$$

In words, $\alpha$ is $t$-satisfied for $u$ in the above sense iff $\alpha$ is satisfied for every object $u'$ and each $t' \in [0, 1]$ such that $t' \geq t$ implies the fact that $\mathit{\Gamma} u'$ is included in $\mathit{\Gamma} u$ in degree at least $t'$. It turns out that

$$u \models^W_t \alpha \text{ iff } \forall u' \in U.(\kappa(\mathit{\Gamma} u', \mathit{\Gamma} u) \geq t \Rightarrow u' \models \alpha) \text{ iff } \mathrm{pos}_t \mathit{\Gamma} u \subseteq ||\alpha||. \tag{11}$$

Observe also that $u \models_1 \alpha$ implies $u \models^W_1 \alpha$.

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[5] The image given by the corresponding uncertainty mapping $\mathit{\Gamma}_{\varrho}$ is a partition of $U$.

[6] Note that $\mathrm{upp}^{\cup} = \mathrm{upp}$ in the case considered by Pawlak.

In our approach, meanings of formulas are granules of information, viz., sets of objects of an AS. They form a system of granules. The recipe for constructing a meaning may be simple just as well as very complex. In the latter case, a number of meanings may be combined in a new, compound one. For example, a meaning of $\alpha$ may be defined as the set $(\mathrm{pos}_{t_1} \circ \mathrm{upp} \circ \mathrm{pos}_{t_2})||\alpha||$, where $t_1, t_2 \in [0, 1]$.

*Example 1.* Consider an AS $\mathcal{M}$, where $U = \{1, \ldots, 5\}$, $\kappa = \kappa^{\pounds}$, $\Gamma 1 = \{1, 2\}$, $\Gamma 2 = \{2, 3, 5\}$, $\Gamma 3 = \{1, 3\}$, $\Gamma 4 = \{4, 5\}$, and $\Gamma 5 = \{2, 4, 5\}$. Let $\alpha$ be a formula such that $||\alpha|| = \{1, 3\}$. Observe that $\mathrm{low}||\alpha|| = \{3\}$, $\mathrm{low}^{\cup}||\alpha|| = \{1, 3\}$, $\mathrm{upp}||\alpha|| = \{1, 2, 3\}$, and $||\alpha||^P = \mathrm{upp}^{\cup}||\alpha|| = \{1, 2, 3, 5\}$. In Table 1, the $t$-positive regions of $\Gamma u$, $||\alpha||_t$, $||\alpha||_{c+t}$, and $||\alpha||_t^W$ are given for various values of $t \in [0, 1]$.

**Table 1.**

| $t$ | $\{0\}$ | $(0, \frac{1}{3}]$ | $(\frac{1}{3}, \frac{1}{2}]$ | $(\frac{1}{2}, \frac{2}{3}]$ | $(\frac{2}{3}, 1]$ |
|---|---|---|---|---|---|
| $\|\|\alpha\|\|_t$ | $U$ | $\{1, 2, 3\}$ | $\{1, 3\}$ | $\{3\}$ | $\{3\}$ |
| $\|\|\alpha\|\|_{c+t}$ | $\{1, 3\}$ | $\{1, 3\}$ | $\{1, 3\}$ | $\{3\}$ | $\{3\}$ |
| $\mathrm{pos}_t \Gamma 1$ | $U$ | $\{1, 2, 3, 5\}$ | $\{1, 3\}$ | $\{1\}$ | $\{1\}$ |
| $\mathrm{pos}_t \Gamma 2$ | $U$ | $U$ | $U$ | $\{2, 5\}$ | $\{2\}$ |
| $\mathrm{pos}_t \Gamma 3$ | $U$ | $\{1, 2, 3\}$ | $\{1, 3\}$ | $\{3\}$ | $\{3\}$ |
| $\mathrm{pos}_t \Gamma 4$ | $U$ | $\{2, 4, 5\}$ | $\{4, 5\}$ | $\{4, 5\}$ | $\{4\}$ |
| $\mathrm{pos}_t \Gamma 5$ | $U$ | $U$ | $\{1, 2, 4, 5\}$ | $\{2, 4, 5\}$ | $\{4, 5\}$ |
| $\|\|\alpha\|\|_t^W$ | $\emptyset$ | $\emptyset$ | $\{1, 3\}$ | $\{1, 3\}$ | $\{1, 3\}$ |

## 4 The Case of Sets of Formulas

Regardless the underlying logic and the notion of satisfiability of formulas for objects, the classical definition of satisfiability of a set of formulas for an object claims that each element of the set be satisfied for the object in the considered sense. In searching for concepts of satisfiability in ASs, serving particular purposes, we go beyond this traditional approach. Like previously, a great variety of interesting notions of satisfiability of sets of formulas may be obtained. Clearly, no universal schema exists, and we discuss a few cases only. Like in the case of single formulas, satisfiability and meaning of sets of formulas are two faces of the same coin: Given a notion of satisfiability, we can derive the corresponding concept of meaning and vice versa, every notion of meaning induces a corresponding notion of satisfiability.

In the approximate case – unlike in the classical one – a set of formulas $X$ may be viewed as satisfied for an object $u$ even if some elements of $X$ are not satisfied for $u$ and, moreover, the notion of satisfiability of formulas may vary from formula to formula. Below, a few classes of special cases are described. In practice, arbitrary combinations of the forms presented, and possibly other ones, may occur.

(A) Firstly, the meaning of $X$ may be defined as a set-theoretical combination of meanings of formulas of a subset of $X$. In particular, the classical case is obtained. For example, assume that only formulas of $X_1 \cup X_2 \cup X_3 = X' \subseteq X$ contribute to the meaning of $X$. For each $\alpha \in X'$, choose a form of satisfiability (or meaning), e.g., $\sigma_\alpha$ ($||\cdot||_\alpha$). Now, let $X$ be satisfied for $u$ iff $\forall\alpha \in X_1.(u,\alpha) \in \sigma_\alpha$ or ($\forall\alpha \in X_2.(u,\alpha) \in \sigma_\alpha$ and $\exists\alpha \in X_3.(u,\alpha) \notin \sigma_\alpha$). Equivalently, the meaning of $X$ may be defined as the set

$$\bigcap_{\alpha\in X_1} ||\alpha||_\alpha \cup \left( \bigcap_{\alpha\in X_2} ||\alpha||_\alpha \cap (U - \bigcap_{\alpha\in X_3} ||\alpha||_\alpha) \right). \tag{12}$$

(B) Next, generalized quantifiers of the form "at most (least) $n$", "more (less) than $n$", "many", "few", "some", etc. may be used in definitions of satisfiability of $X$ for $u$. For instance, one may require that $X$ be satisfied for $u$ iff at least $m$ formulas of $X$ are satisfied for $u$ and at most $n$ formulas of $X$ are unsatisfied for $u$, where $m, n$ are natural numbers.

(C) Generalized quantifiers may also be used to speak of relative quantities (frequencies, percentages). For example, one may stipulate that $X$ be satisfied for $u$ iff at least $m\%$ of formulas of $X_1 \subseteq X$ are satisfied for $u$ and less than $m'\%$ of formulas of $X_2 \subseteq X$ are unsatisfied for $u$ and there is an object $u'$, similar to $u$, for which more than $n\%$ formulas of $X_3 \subseteq X$ are satisfied. Here $m, m', n$ are reals from $[0, 100]$. Cases described by (A) may easily be formalized within (C). In particular, the classical case is obtained by claiming that 100% of formulas of $X$ be satisfied for $u$. Also case (B) may be formalized in terms of frequencies if $X$ is finite.

(D) Instead of speaking of a percentage of formulas to be satisfied, we may refer to probabilities. As an example we can take the following definition schema: $X$ is satisfied for $u$ iff the probability that a formula $\alpha \in X$ is unsatisfied in a given sense is less than $p$.

In what follows, we focus upon a formal description of case (C). Consider a RIF $\kappa^* : (\wp\mathrm{FOR})^2 \mapsto [0, 1]$ and a non-empty set of formulas $X$ of L. For every $\alpha \in X$, choose a relation of satisfiability $\sigma_\alpha$ (or a mapping of meaning $||\cdot||_\alpha$). Next, divide $X$ into parts – non-empty subsets $X'$ such that for any $\alpha, \beta \in X'$, $\sigma_\alpha = \sigma_\beta$. For simplicity, assume that a partition $\mathcal{X} = \{X_i\}_{i\in I}$ as above is given and $\sigma_i$ is a satisfiability relation associated with $X_i$. For every $X_i$, choose a mapping $f_i : \wp U \mapsto \wp U$ (e.g., an approximation mapping) and set a threshold value $t_i \in [0, 1]$. Recall that $|u|_\sigma$ denotes the set of formulas satisfied for $u$ in the sense of $\sigma$. We can measure the degree of rough inclusion of $X_i$ in $|u|_{\sigma_i}$. If $\kappa^*$ is standard, $k_i = \kappa^*(X_i, |u|_{\sigma_i})$ is the fraction of formulas of $X$ which are $\sigma_i$-satisfied for $u$. Let

$$||X_i||_{\geq t_i} = \{u \mid k_i \geq t_i\} \text{ and } ||X_i||_{>t_i} = \{u \mid k_i > t_i\}. \tag{13}$$

Additionally, $||X_i||_{\leq t_i} = U - ||X_i||_{>t_i}$, $||X_i||_{<t_i} = U - ||X_i||_{\geq t_i}$, and $||X_i||_{=t_i} = \{u \mid k_i = t_i\}$. By the meaning of $X_i$ we understand $||X_i|| = f_i Y$, where $Y$ is

among $||X_i||_{\geq t_i}, ||X_i||_{\leq t_i}, ||X_i||_{>t_i}, ||X_i||_{<t_i}, ||X_i||_{=t_i}$. Then, the meaning of $X$, $||X||$, is a set-theoretical combination of meanings of parts of $X$. Finally, we say that $X$ is satisfied for $u$ iff $u \in ||X||$.

The schema above is fairly general and many interesting cases may be obtained. In particular, $||X_i||_{\geq 1}$ consists of all objects for which all formulas of $X_i$ are $\sigma_i$-satisfied, $||X_i||_{\leq 0}$ is the set of all objects for which no formula of $X_i$ is $\sigma_i$-satisfied, $||X_i||_{\leq 1} = U = ||X_i||_{\geq 0}$, and $||X_i||_{>1} = \emptyset = ||X_i||_{<0}$.

All formulas of $X$ to be omitted may be collected into one set, say $Y$. It suffices to choose $\models_0$ as the satisfiability relation (since $||\alpha||_0 = U$ for any $\alpha$), the identity mapping as $f$, and any $t \in [0,1]$ as the threshold. Then, $||Y|| = U$ and for any set $Z \subseteq U$, $||Y|| \cap Z = Z$.

By introducing mappings $f$, various forms of reasoning by analogy may be modelled. For instance, if $||X_i|| = \mathrm{upp}||X_i||_{\geq t}$, then $X_i$ is satisfied for $u$ iff there is an object $u'$ similar to $u$ such that $u' \in ||X_i||_{\geq t}$. Thus, we reason about satisfiability of $X_i$ for $u$ on the base of some form of satisfiability of this set for some object similar to $u$.

Henceforth, $u \models \alpha$ and $||\alpha||$ will also be written as $u \models_c \alpha$ and $||\alpha||_c$, respectively. Next, let $s \in T = [0,1] \cup \{c\}$. The graded forms of satisfiability and meaning of a set of formulas, introduced and studied in [7], are obtained by taking $\mathcal{X} = \{X\}$, $\sigma = \models_s$, the identity mapping as $f$, and $||X||_{\geq t}$ as the meaning of $X$. Thus, all formulas of $X$ are uniformly treated. According to our original notation, $X$ is $(s,t)$-*satisfied* for $u$ iff $\kappa^*(X, |u|_s) \geq t$, i.e., iff a sufficiently large part of $X$ is satisfied in a sufficient degree, where sufficiency is determined by $s$ and $t$. The corresponding meaning of $X$, $||X||_{\geq t}$, is denoted by $||X||_{(s,t)}$ here.

Within this formalism we can model the situation that formulas of a set are of various preference (importance, etc.) like, e.g., premises in default rules [19]. To this end, consider a (pre)ordering relation $<$ on the set $I$ of indexes of the partition $\mathcal{X}$ of a set of formulas $X$. $i < j$ means that the class $X_j$ is of higher priority than the class $X_i$. For each $i \in I$, choose a fundamental graded satisfiability relation $\models_{s_i}$ as $\sigma_i$ in such a way that $s_i < s_j$ if $i < j$, and $s_i = s_j$ if $i, j$ are equally preferred. Select threshold values $t_i$ appropriately. Finally, define the meaning of $X$ as the intersection $\bigcap\{||X_i||_{(s_i,t_i)} \mid i \in I\}$.

Consider the case $I = \{1, 2\}$. We can easily formalize the idea that a set of formulas $X$ is satisfied for $u$ iff sufficiently many formulas of lesser priority (importance) are possibly satisfied and sufficiently many formulas of greater priority (importance) are certainly satisfied for $u$, where sufficiency is determined by some threshold values as earlier. Suppose that $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$, and that formulas of $X_1$ are of lesser priority than those from $X_2$. For any $\alpha$ and $u$, let $(u, \alpha) \in \sigma_1$ iff $u \in \mathrm{upp}||\alpha||$ and $(u, \alpha) \in \sigma_2$ iff $u \in \mathrm{low}||\alpha||$. Given $t_1, t_2 \in [0,1]$, define the meaning of $X$ as the intersection $||X_1||_{\geq t_1} \cap ||X_2||_{\geq t_2}$.

*Example 2.* Let $X = \{\alpha_1, \ldots, \alpha_9\}$ be a set of formulas partitioned into three components: $X_1 = \{\alpha_1, \alpha_2, \alpha_3\}$, $X_2 = \{\alpha_4, \ldots, \alpha_7\}$, and $X_3 = \{\alpha_8, \alpha_9\}$. All elements of $X_i$ $(i = 1, 2, 3)$ are treated equally with respect to satisfiability. Assume that the RIF $\kappa^*$ is standard. Next, let the corresponding forms of satisfiability of formulas be $\sigma_1 = \models_1$, $\sigma_2 = \models_{0.5}$, and $\sigma_3 = \models_c$. As the meanings of $X_1, X_2, X_3$, and $X$, we take $||X_1||_{\geq 1}$, $||X_2||_{\geq 0.75}$, $\mathrm{upp}||X_3||_{\geq 0.5}$, and the intersection

$$||X_1||_{\geq 1} \cap ||X_2||_{\geq 0.75} \cap \mathrm{upp}||X_3||_{\geq 0.5},$$

respectively. That is, $X$ is satisfied for an object $u$ iff all formulas of $X_1$ are 1-satisfied for $u$, at least 75% of formulas of $X_2$ are 0.5-satisfied for $u$, and there is an object, similar to $u$, for which at least 50% of formulas of $X_3$ are satisfied (in the crisp sense). Recall that a formula $\alpha$ is 1-satisfied for $u$ iff $\alpha$ is satisfied for all objects similar to $u$. Analogously, if $\kappa$ is quasi-standard and $\Gamma u$ is finite, then $\alpha$ is 0.5-satisfied for $u$ iff $\alpha$ is satisfied for at least 50% of objects similar to $u$. Let $4 \leq j \leq 7$. It turns out that

$$||X_1||_{\geq 1} = ||X_1||_{(1,1)} = \mathrm{low} \bigcap_{\alpha \in X_1} ||\alpha|| = \mathrm{low}||\alpha_1|| \cap \mathrm{low}||\alpha_2|| \cap \mathrm{low}||\alpha_3||;$$

$$||X_2||_{\geq 0.75} = \bigcap_{j \neq 4} ||\alpha_j||_{0.5} \cup \bigcap_{j \neq 5} ||\alpha_j||_{0.5} \cup \bigcap_{j \neq 6} ||\alpha_j||_{0.5} \cup \bigcap_{j \neq 7} ||\alpha_j||_{0.5};$$

$$\mathrm{upp}||X_3||_{\geq 0.5} = \mathrm{upp} \bigcup_{\alpha \in X_3} ||\alpha|| = \mathrm{upp}||\alpha_8|| \cup \mathrm{upp}||\alpha_9||.$$

## 5 A Note on Meaning and Applicability of Rules

In this article, the problem of meaning and applicability of rules is merely touched upon. In [6, 3], we propose and study several various forms of applicability of rules. Rules over L, denoted by $r$ with subscripts if needed, are pairs of finite sets of formulas of L. Any such rule $r$ is of the form $(P_r, C_r)$, where $P_r$ is the set of *premises* and $C_r$ – the set of *conclusions* of $r$. By assumption, the latter set is non-empty. Briefly speaking, the meaning of a rule $r$ may be defined as a pair of meanings of the sets of premises and conclusions of $r$, respectively. Since premises and conclusions of $r$ may be understood in a different way, we start with two mappings of meaning $|| \cdot ||_1, || \cdot ||_2 : \wp\mathrm{FOR} \mapsto \wp U$ and define the *meaning* of $r$ as the pair

$$||r||_{1,2} = (||P_r||_1, ||C_r||_2). \tag{14}$$

Next, we can say that $r$ is *applicable* (resp., *applicable to* an object $u$) iff $||P_r||_1 \neq \emptyset$ (resp., $u \in ||P_r||_1$), i.e., iff $P_r$ is satisfiable (satisfied for $u$) in the sense of $|| \cdot ||_1$. In the real life, understanding and application of rules are context-dependent. They often vary from agent to agent, from situation to situation, from rule to rule, etc. Hence, the notions of meaning and applicability should be equipped with lists of tuning parameters for the sake of a better adaptivity to a particular case.

As an example of applicability of rules, let us recall the notion of a *graded applicability*, introduced in [6] and next generalized in [3]. For any $t \in T_1$, a rule $r$ is said to be *t-applicable* (resp., *t-applicable to u*) iff $||P_r||_t \neq \emptyset$ (resp., $u \in ||P_r||_t$), where $||P_r||_t$ is the graded $t$-meaning of $P_r$, defined in [7] and recalled in the preceding section.

A better understanding of various forms of applicability of rules in ASs and the relationships among them is important, e.g., for the purpose of modeling and a formal analysis of systems of social agents.

*Example 3.* Consider a system of deliberative cooperating agents (e.g., an examining board) who, for simplicity, speak the same language and have the same knowledge base in the form of an AS. The agents are allowed to understand formulas, sets of formulas, or rules differently. They make collective decisions by voting, where more than 50% of agents have to agree upon a question to arrive at a decision. Clearly, the effectiveness of interaction of the system heavily depends on whether or not the agents' notions of satisfiability/meaning are compatible with one another. In the case of two agents, the process of decision making will be blocked if every time a decision rule $r$ is applicable for one agent, it is inapplicable for the latter one.

## 6 Concluding Remarks

In this article, intended as a concise study of the general notions of satisfiability and meaning of formulas and their sets within the framework of ASs, a few concepts of satisfiability/meaning, already known, have been recalled as well as some new ones have been proposed. Many of them seem to be suitable for modeling of various forms of reasoning about the fact that a formula/set of formulas hold for some object or that a rule is applicable to an object, and subsequently, for modeling of systems of adaptive social agents.

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