Towards Rough Applicability of Rules

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Summary. In this article, we further study the problem of soft applicability of rules within the framework of approximation spaces. Such forms of applicability are generally called *rough*. The starting point is the notion of graded applicability of a rule to an object, introduced in our previous work and referred to as *fundamental*. The abstract concept of rough applicability of rules comprises a vast number of particular cases. In the present paper, we generalize the fundamental form of applicability in two ways. Firstly, we more intensively exploit the idea of rough approximation of sets of objects. Secondly, a graded applicability of a rule to a set of objects is defined. A better understanding of rough applicability of rules is important for building the ontology of an approximate reason and, in the sequel, for modeling of complex systems, e.g., systems of social agents.

Key words: approximation space, ontology of approximate reason, information granule, graded meaning of formulas, applicability of rules

To Emilia

1 Introduction

It is hardly an exaggeration to say that soft application of rules is the prevailing form of rule following in real life situations. Though some rules (e.g., instructions, regulations, laws, etc.) are supposed to be strictly followed, it usually means “as strictly as possible” in practice. Typically, people tend to apply rules “softly” whenever the expected advantages (gain) surpass the possible loss (failure, harm). Soft application of rules is usually more efficient and

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effective than the strict one, however, at the cost of the results obtained. In many cases, adaptation to changing situations requires a change in the mode of application of rules only, retaining the rules unchanged. Allowing rules to be applied softly simplifies multi-attribute decision making under missing or uncertain information as well.

As a research problem, applicability of rules concerns strategies (meta-rules) which specify the permissive conditions for passing from premises to conclusions of rules. In this paper, we analyze soft applicability of rules within the framework of approximation spaces (ASs) or, in other words, rough applicability of rules. The first step has been already made by introducing the concept of graded applicability of a rule to an object of an AS [3]. This *fundamental* form of applicability is based on the graded satisfiability and meaning of formulas and their sets, studied in [2]. The intuitive idea is that a rule r is applicable to an object u in degree t iff a sufficiently large part of the set of premises of r is satisfied for u in a sufficient degree, where sufficiency is determined by t. We aim at extending and refining this notion step by step. For the time being, we propose two generalizations. In the first one, the idea of approximation of sets of objects is exploited more intensively. The second approach consists in extending the graded applicability of a rule to an object to the case of graded applicability of a rule to a set of objects.

Studying various rough forms of applicability of rules is important for building the *ontology of an approximate reason*. In [9], Peters et al. consider structural aspects of such an ontology. A basic assumption made is that an approximate reason is a capability of an agent. Agents classify information granules, derived from sensors or received from other agents, in the context of ASs. One of the fundamental forms of reasoning is a reflective judgment that a particular object (granule of information) matches a particular pattern. In the case of rules, agents judge whether or not, and how far an object (set of objects) matches the conditions for applicability of a rule. As explained in [9]:

> Judgment in agents is a faculty of thinking about (classifying) the particular relative to decision rules derived from data. Judgment in agents is reflective but not in the classical philosophical sense [...]. In an agent, a reflective judgment itself is an assertion that a particular decision rule derived from data is applicable to an object (input). [...] Again, unlike Kant's notion of judgment, a reflective judgment is not the result of searching for a universal that pertains to a particular set of values of descriptors. Rather, a reflective judgment by an agent is a form of recognition that a particular vector of sensor values pertains to a particular rule in some degree.

The ontology of an approximate reason may serve as a basis for modeling of complex systems like systems of social, highly adaptive agents, where rules are allowed to be followed flexibly and approximately. Since one and the same rule may be applied in many ways depending, among others, on the agent and the situation of (inter)action, we can to a higher extent capture the complexity of the modelled system by means of relatively less rules. Moreover, agents are given more autonomy in applying rules. From the technical point of view, degrees of applicability may serve as lists of tuning parameters to control application of rules. Another area of possible use of rough applicability is multi-attribute classification (and, in particular, decision making). In the case of an object to which no classification rule is applicable in the strict sense, we may try to apply an available rule roughly. This happens in the real life, e.g., in the process of selection of the best candidate(s), where no candidate fully satisfies the requirements. If a decision is to be made anyway, some conditions should be omitted or their satisfiability should be treated less strictly. Rough applicability may also help in classification of objects, where some values of attributes are missing.

In Sect. 2, approximation spaces are overviewed. Section 3 is devoted to the notions of graded satisfiability and meaning of formulas. In Sect. 4, we generalize the fundamental notion of applicability in the two directions mentioned earlier. Section 5 contains a concise summary.

2 Approximation Spaces

The general notion of an *approximation space* (AS) was proposed by Skowron and Stepaniuk [13, 14, 16]. Any such space is a triple $\mathcal{M} = (U, \Gamma, \kappa)$, where U is a non-empty set, $\Gamma : U \mapsto \wp U$ is an *uncertainty mapping*, and $\kappa : (\wp U)^2 \mapsto [0, 1]$ is a *rough inclusion function* (RIF). $\wp U$ and $(\wp U)^2$ denote the power set of U and the Cartesian product $\wp U \times \wp U$, respectively. Originally, Γ and κ were equipped with tuning parameters, and the term "parameterized" was therefore used in connection with ASs. Exemplary ASs are the rough ASs, induced by the *Pawlak information systems* [6, 8].

Elements of U, called objects and denoted by u with subscripts whenever needed, are known by their properties only. Therefore, some objects may be viewed as similar. Objects similar to an object u constitute a *granule of information* in the sense of Zadeh [17]. Indiscernibility may be seen as a special case of similarity. Since every object is obviously similar to itself, the universe U of $\mathcal{M}$ is covered by a family of granules of information. The uncertainty mapping Γ is a basic mathematical tool to describe formally granulation of information on U. For every object u, Γu is a set of objects similar to u, called an *elementary* granule of information drawn to u. By assumption, $u \in \Gamma u$. Elementary granules are merely building blocks to construct more complex information granules which form, possibly hierarchical, systems of granules. Simple examples of complex granules are the results of set-theoretical operations on granules obtained at some earlier stages, rough approximations of concepts, or meanings of formulas and sets of formulas in ASs. An adaptive calculus of granules, measure(s) of closeness and inclusion of granules, construction of complex granules from simpler ones which satisfy a given specification are a few examples of related problems (see, e.g., [11, 12, 15, 16]).

In our approach, a RIF $\kappa : (\wp U)^2 \mapsto [0,1]$ is a function which assigns to every pair (x,y) of subsets of U, a number in $[0,1]$ expressing the degree of inclusion of x in y, and which satisfies postulates (A1)–(A3) for any $x, y, z \subseteq U$: (A1) $\kappa(x,y) = 1$ iff $x \subseteq y$; (A2) If $x \neq \emptyset$, then $\kappa(x,y) = 0$ iff $x \cap y = \emptyset$; (A3) If $y \subseteq z$, then $\kappa(x,y) \leq \kappa(x,z)$. Thus, our RIFs are somewhat stronger than the ones characterized by the axioms of *rough mereology*, proposed by Polkowski and Skowron [10, 12]. Rough mereology extends Leśniewski's mereology [4] to a theory of the relationship of being-a-part-in-degree.

Among various RIFs, the *standard* ones deserve a special attention. Let the cardinality of a set x be denoted by $\#x$. Given a non-empty finite set U and $x, y \subseteq U$, the standard RIF, $\kappa^{\pounds}$, is defined by $\kappa^{\pounds}(x,y) = \begin{cases} \frac{\#(x \cap y)}{\#x} & \text{if } x \neq \emptyset \\ 1 & \text{otherwise.} \end{cases}$ The notion of a standard RIF, based on the frequency count, goes back to Łukasiewicz [5]. In our framework, where infinite sets of objects are allowed, by a *quasi-standard* RIF we understand any RIF which for finite first arguments is like the standard one.

In $\mathcal{M}$, sets of objects (concepts) may be approximated in various ways (see, e.g., [1] for a discussion and references). In [14, 16], a concept $x \subseteq U$ is approximated by means of the *lower* and *upper rough approximation mappings* $\mathrm{low}, \mathrm{upp} : \wp U \mapsto \wp U$, respectively, defined by

$$\mathrm{low}x = \{u \in U \mid \kappa(\Gamma u, x) = 1\} \text{ and } \mathrm{upp}x = \{u \in U \mid \kappa(\Gamma u, x) > 0\}. \quad (1)$$

By (A1)–(A3), the lower and upper rough approximations of x, $\mathrm{low}x$ and $\mathrm{upp}x$, are equal to $\{u \in U \mid \Gamma u \subseteq x\}$ and $\{u \in U \mid \Gamma u \cap x \neq \emptyset\}$, respectively.

Ziarko [18, 19] generalized the Pawlak rough set model [7, 8] to a variable-precision rough set model by introducing variable-precision positive and negative regions of sets of objects. Let $t \in [0,1]$. Within the AS framework, in line with (1), the mappings of *t-positive* and *t-negative regions* of sets of objects, $\mathrm{pos}_t, \mathrm{neg}_t : \wp U \mapsto \wp U$, respectively, may be defined as follows, for any set of objects x:[2]

$$\mathrm{pos}_t x = \{u \in U \mid \kappa(\Gamma u, x) \geq t\} \text{ and } \mathrm{neg}_t x = \{u \in U \mid \kappa(\Gamma u, x) \leq t\}. \quad (2)$$

Notice that $\mathrm{low}x = \mathrm{pos}_1 x$ and $\mathrm{upp}x = U - \mathrm{neg}_0 x$.

3 The Graded Meaning of Formulas

Suppose a formal language L expressing properties of $\mathcal{M}$ is given. The set of all formulas of L is denoted by FOR. We briefly recall basic ideas concerning the graded satisfiability and meaning of formulas and their sets, studied in [2]. Given a relation of (crisp) satisfiability of formulas for objects of U, $\models_c$, the *c-meaning* (or, simply, *meaning*) of a formula α is understood as the extension

[2] The original definitions, proposed by Ziarko, are somewhat different.

of α, i.e., as the set $||\alpha||_c = \{u \in U \mid u \models_c \alpha\}$. For simplicity, "$c$" will be omitted in formulas whenever possible. By introducing degrees $t \in [0,1]$, we take into account the fact that objects are perceived through the granules of information attached to them. In the formulas below, $u \models_t \alpha$ reads as "α is *t-satisfied* for u" and $||\alpha||_t$ denotes the *t-meaning* of α:

$$u \models_t \alpha \text{ iff } \kappa(\Gamma u, ||\alpha||) \geq t \text{ and } ||\alpha||_t = \{u \in U \mid u \models_t \alpha\}. \tag{3}$$

In other words, $||\alpha||_t = \mathrm{pos}_t||\alpha||$. Next, for $t \in T = [0,1] \cup \{c\}$, the set of all formulas which are t-satisfied for an object u is denoted by $|u|_t$, i.e., $|u|_t = \{\alpha \in \mathrm{FOR} \mid u \models_t \alpha\}$. Notice that it may be $t = c$ here.

The graded satisfiability of a formula for an object is generalized on the left-hand side to a graded satisfiability of a formula for a set of objects, and on the right-hand side to a graded satisfiability of a set of formulas for an object, where degrees are elements of $T_1 = T \times [0,1]$. For any n-tuple t and $i = 1, \ldots, n$, let $\pi_i t$ denote the i-th element of t. For simplicity, we use $\models_t$, $|\cdot|_t$, and $||\cdot||_t$ both for the (object, formula)-case as well as for its generalizations. Thus, for any object u, a set of objects x, a formula α, a set of formulas X, a RIF $\kappa^* : (\wp\mathrm{FOR})^2 \mapsto [0,1]$, and $t \in T_1$,

$$\begin{aligned} &x \models_t \alpha \text{ iff } \kappa(x, ||\alpha||_{\pi_1 t}) \geq \pi_2 t \text{ and } |x|_t = \{\alpha \in \mathrm{FOR} \mid x \models_t \alpha\}; \\ &u \models_t X \text{ iff } \kappa^*(X, |u|_{\pi_1 t}) \geq \pi_2 t \text{ and } ||X||_t = \{u \in U \mid u \models_t X\}. \end{aligned} \tag{4}$$

$u \models_t X$ reads as "X is *t-satisfied* for u", and $||X||_t$ is the *t-meaning* of X. Observe that $\models_t$ extends the classical, crisp notions of satisfiability of the sorts (set-of-objects, formula) and (object, set-of-formulas). Along the standard lines, $x \models \alpha$ iff $\forall u \in x.u \models \alpha$, and $u \models X$ iff $\forall \alpha \in X.u \models \alpha$. Hence, $x \models \alpha$ iff $x \models_{(c,1)} \alpha$, and $u \models X$ iff $u \models_{(c,1)} X$. Properties of the graded satisfiability and meaning of formulas and sets of formulas may be found in [2]. Let us only mention that a non-empty finite set of formulas X cannot be replaced by a conjunction $\bigwedge X$ of all its elements as it happens in the classical, crisp case. In the graded case, one can only prove that $||\bigwedge X||_t \subseteq ||X||_{(t,1)}$, where $t \in T$, but the converse may not hold.

4 The Graded Applicability of Rules Generalized

All rules over L, denoted by r with subscripts whenever needed, constitute a set RUL. Any rule r is a pair of finite sets of formulas of L, where the first element, P_r, is the set of *premises* of r and the second element of the pair is a non-empty set of *conclusions* of r. Along the standard lines, a rule which is not applicable in a considered sense is called inapplicable.

A rule r is applicable to an object u in the classical sense iff the whole set of premises P_r is satisfied for u. The graded applicability of a rule to an object, viewed as a fundamental form of rough applicability here, is obtained by replacing the crisp satisfiability by its graded counterpart and by weakening the condition that all premises be satisfied [3]. Thus, for any $t \in T_1$,

$$r \in \mathrm{apl}_t u \ \text{ iff } \ \kappa^*(P_r, |u|_{\pi_1 t}) \geq \pi_2 t, \ \text{ i.e., iff } \ u \in ||P_r||_t. \tag{5}$$

$r \in \mathrm{apl}_t u$ reads as "r is *t-applicable* to u".[3] Properties of apl_t are presented in [3]. Let us only note that the classical applicability and the $(c, 1)$-applicability coincide.

Example 1. In the textile industry, a norm determining whether or not the quality of water to be used in the process of dyeing of textiles is satisfactory, may be written as a decision rule r with 16 premises and one conclusion (d, yes). In this case, the objects of the AS considered are samples of water. The c-meaning of the conclusion of r is the set of all samples of water $u \in U$ such that the water may be used for dyeing of textiles, i.e., $||(d, \mathrm{yes})|| = \{u \in U \mid d(u) = \mathrm{yes}\}$. Let $a_1, \ldots, a_7$ denote the attributes: colour (mg Pt/l), turbidity (mg SiO_2/l), suspensions (mg/l), oxygen consumption (mg O_2/l), hardness (mval/l), Fe content (mg/l), and Mn content (mg/l), respectively. Then, $(a_1, [0, 20])$, $(a_2, [0, 15])$, $(a_3, [0, 20])$, $(a_4, [0, 20])$, $(a_5, [0, 1.8])$, $(a_6, [0, 0.1])$, and $(a_7, [0, 0.05])$ are exemplary premises of r. For instance, the c-meaning of $(a_2, [0, 15])$ is the set of all samples of water such that their turbidity does not exceed 15 mg SiO_2/l, i.e., $||(a_2, [0, 15])|| = \{u \in U \mid a_2(u) \leq 15\}$. Suppose that the values of a_2, a_3 slightly exceed $15, 20$ for some sample u, respectively, i.e., the second and the third premises are not satisfied for u, whereas all remaining premises hold for u. That is, r is inapplicable to the sample u in the classical sense, yet it is $(c, 0.875)$-applicable to u. Under special conditions as, e.g., serious time constraints, applicability of r to u in degree $(c, 0.875)$ may be viewed as sufficient or, in other words, the quality of u may be viewed as satisfactory if the gain expected surpass the possible loss.

Observe that $r \in \mathrm{apl}_t u$ iff $u \in I_{\wp U}||P_r||_t$, where $I_{\wp U}$ is the identity mapping on $\wp U$. A natural generalization of (5) is obtained by taking a mapping $f_\$: \wp U \mapsto \wp U$ instead of $I_{\wp U}$, where \$ is a possibly empty list of parameters. For instance, $f_\$$ may be an approximation mapping. In this way, we obtain a family of mappings $\mathrm{apl}_t^{f_\$} : U \mapsto \wp\mathrm{RUL}$, parameterized by $t \in T_1$ and \$, and such that for any r and u,

$$r \in \mathrm{apl}_t^{f_\$} u \overset{\mathrm{def}}{\leftrightarrow} u \in f_\$||P_r||_t. \tag{6}$$

The family is partially ordered by $\sqsubseteq$, where for any $t_1, t_2 \in T_1$,

$$\mathrm{apl}_{t_1}^{f_\$} \sqsubseteq \mathrm{apl}_{t_2}^{f_\$} \overset{\mathrm{def}}{\leftrightarrow} \forall u \in U. \mathrm{apl}_{t_1}^{f_\$} u \subseteq \mathrm{apl}_{t_2}^{f_\$} u. \tag{7}$$

The general notion of rough applicability, introduced above, comprises a number of particular cases, including the fundamental one. In fact, $\mathrm{apl}_t =$

[3] Equivalently, "r is *applicable* to u *in degree* t".

$\mathrm{apl}_t^{I_{\wp U}}$. Next, e.g., $r \in \mathrm{apl}_t^{\mathrm{low}}u$ iff $u \in \mathrm{low}||P_r||_t$ iff r is t-applicable to every object similar to u. In the same vein, $r \in \mathrm{apl}_t^{\mathrm{upp}}u$ iff $u \in \mathrm{upp}||P_r||_t$ iff r is t-applicable to some object similar to u. We can also say that r is *certainly* t-applicable and *possibly* t-applicable to u, respectively. In the variable-precision case, for $f = \mathrm{pos}_s$ and $s \in [0,1]$, $r \in \mathrm{apl}_t^f u$ iff $u \in \mathrm{pos}_s||P_r||_t$ iff r is t-applicable to a sufficiently large part of Γu, where sufficiency is determined by s. In a more sophisticated case, where $f = \mathrm{pos}_s \circ \mathrm{low}$ ($\circ$ denotes the concatenation of mappings), $r \in \mathrm{apl}_t^f u$ iff $u \in \mathrm{pos}_s\mathrm{low}||P_r||_t$ iff $\kappa(\Gamma u, \mathrm{low}||P_r||_t) \geq s$ iff r is certainly t-applicable to a sufficiently large part of Γu, where sufficiency is determined by s. Etc.

For $t = (t_1, t_2) \in [0,1]^2$, the various forms of rough t-applicability are determined up to granularity of information. An object u is merely viewed as a representative of the granule of information Γu drawn to it. More precisely, a rule r may practically be treated as applicable to u even if no premise is, in fact, satisfied for u. It is enough that premises are satisfiable for a sufficiently large part of the set of objects similar to u. If used reasonably, this feature may be advantageous in the case of missing data. The very idea is intensified in the case of pos_s. Then, r is t-applicable to u in the sense of pos_s iff it is t-applicable to a sufficiently large part of the set of objects similar to u, where sufficiency is determined by s. This form of applicability may be helpful in classification of u if we cannot check whether or not r is applicable to u and, on the other hand, it is known that r is applicable to a sufficiently large part of the set of objects similar to u. Next, rough applicability in the sense of low is useful in modeling of such situations, where the stress is laid on the equal treatment of all objects forming a granule of information. A form of stability of rules may be defined, where r is called *stable* in a sense considered if for every u, r is applicable to u iff r is applicable to all objects similar to u in the very sense.

Example 2. Consider a situation of decision making whether or not to support a student financially. In this case, objects of the AS are students applying for a bursary. Suppose that some data concerning a person u is missing which makes decision rules inapplicable to u in the classical, crisp sense. For simplicity, assume that r would be the only decision rule applicable to u unless the data were missing. Let α be the premise of r of which we cannot be sure if it is satisfied for u or not. Suppose that for 80% of students whose cases are similar to the case of u, all premises of r are satisfied. Then, to the advantage of u, we may view r as practically applicable to u. Formally, r is $(0.8, 1)$-applicable to u. Additionally, let r be $(0.8, 0.9)$-applicable to 65% of objects similar to u. In sum, r is $(0.8, 0.9)$-applicable to u in the sense of $\mathrm{pos}_{0.65}$.

The second (and last) generalization of the fundamental notion of rough aplicability, proposed here, consists in extension of applicability of a rule to an object to the case of applicability of a rule to a set of objects. In the classical case, a rule is applicable to a set of objects x iff it is applicable to each element of x. For any a, let $(a)^n$ denote the tuple consisting of n copies of a, and $(a)^1$ be abbreviated by (a). For arbitrary tuples s, t, st denotes their concatenation. Next, if t is at least a pair of items (i.e., an n-tuple for $n \geq 2$), then $\lhd t$ is the tuple obtained from t by removing the last element. For example, $(a, b)(1) = (a, b, 1)$ and $\lhd(a, b, 1) = (a, b)$. In the graded case, where $t \in T_2 = T_1 \times [0, 1] = T \times [0, 1] \times [0, 1]$, r is called *t-applicable* to x, $r \in \mathrm{Apl}_t x$, iff r is $\lhd t$-applicable to a sufficiently large part of x, where sufficiency is determined by $\pi_3 t$, i.e.,

$$r \in \mathrm{Apl}_t x \overset{\mathrm{def}}{\leftrightarrow} \kappa(x, ||P_r||_{\lhd t}) \geq \pi_3 t. \tag{8}$$

Thus, a family of mappings $\mathrm{Apl}_t : \wp U \mapsto \wp \mathrm{RUL}$ is obtained, parameterized by $t \in T_2$ and partially ordered by a relation $\sqsubseteq$, where for any $t_1, t_2 \in T_2$,

$$\mathrm{Apl}_{t_1} \sqsubseteq \mathrm{Apl}_{t_2} \overset{\mathrm{def}}{\leftrightarrow} \forall x \subseteq U.\mathrm{Apl}_{t_1} x \subseteq \mathrm{Apl}_{t_2} x. \tag{9}$$

The graded applicability, introduced above, is an exemplary notion of rough applicability of a rule to a complex object which is a set of objects of the underlying approximation space $\mathcal{M}$ in our case. This notion may be useful in modeling of a number of situations. Three such cases are sketched below.

Example 3. Suppose that objects of an AS are questions which may be subject to negotiation. Then, sets of objects are packets of such questions and represent possible negotiation problems. Let the rules considered be decision rules on how to solve particular problems. We can rank decision rules depending, among others, on their graded applicability to given negotiation problems. The more questions solved positively by a rule, the better is the rule.

Example 4. Let objects of an AS be school students in a town. A committee constructs rules to rank classes of students in order to award a prize to the best class. They search for the most universal rule(s) satisfying some additional conditions. A rule r is viewed as more universal than a rule r' iff r applies in a considered sense to larger parts of given classes of students than r' does.

Example 5. In a factory, every lot of products is tested whether or not the articles comply with a norm r or, in other words, how far the norm r is applicable in some considered sense to every lot of products. In this case, products are objects of an AS and lots of products are the complex objects considered. A lot x passes the test if a sufficiently large part of x complies with r or, in other words, if r applies to x in a sufficient degree.

Below, we present a number of properties of the forms of applicability of rules defined earlier. For natural numbers $n \geq 1$, $i = 1, \ldots, n$, non-empty partially ordered sets $(x_i, \leq_i)$, and tuples $s, t \in x_1 \times \ldots \times x_n$, let $s \preceq t$ iff $\forall i = 1, \ldots, n.\pi_i s \leq_i \pi_i t$. As usual, $\succeq$ is the converse relation of $\preceq$. The natural total ordering $\leq$ on $[0, 1]$ is extended to a partial ordering on T by taking $c \leq c$. A mapping $f : \wp x \mapsto \wp y$ is *monotone* iff for any $x_1 \subseteq x_2 \subseteq x$, $fx_1 \subseteq fx_2$.

Theorem 1. *For any objects u, u', a set of objects x, a mapping $f_\$: \wp U \mapsto \wp U$, $s \in [0,1]$, and $t, t' \in T_1$, we have:*

(a) Where $f = \mathrm{pos}_s$, $\mathrm{apl}_t^f u = \mathrm{Apl}_{t(s)} \Gamma u$.

(b) $\mathrm{apl}_t^{\mathrm{low}} = \mathrm{apl}_t^{\mathrm{pos}_1}$ and $\mathrm{apl}_t^{\mathrm{upp}} u = \bigcup_{s>0} \{\mathrm{apl}_t^f u \mid f = \mathrm{pos}_s\}$.

(c) If $\Gamma u = \Gamma u'$ and $g \in \{\mathrm{upp} \circ f_\$, \mathrm{pos}_s \circ f_\$\}$, then $\mathrm{apl}_t u = \mathrm{apl}_t u'$ and $\mathrm{apl}_t^g u = \mathrm{apl}_t^g u'$.

(d) If $f_\$$ is monotone and $t \preceq t'$, then $\mathrm{apl}_{t'}^{f_\$} \sqsubseteq \mathrm{apl}_t^{f_\$}$.

(e) $\mathrm{apl}_t^{\mathrm{low}} \sqsubseteq \mathrm{apl}_t \sqsubseteq \mathrm{apl}_t^{\mathrm{upp}}$.

(f) $\mathrm{Apl}_{t(1)} x = \bigcap \{\mathrm{apl}_t u \mid u \in x\}$.

Proof. We prove (d), (f) only. For (d) consider a rule r and assume (d1) $f_\$$ is monotone and (d2) $t \preceq t'$. First, we show (d3) $||P_r||_{t'} \subseteq ||P_r||_t$. Consider the non-trivial case only, where $\pi_1 t, \pi_1 t' \neq c$. Assume that $u \in ||P_r||_{t'}$. Then (d4) $\kappa^*(P_r, |u|_{\pi_1 t'}) \geq \pi_2 t'$ by the definition of graded meaning. Observe that for any formula α, if $\kappa(\Gamma u, ||\alpha||) \geq \pi_1 t'$, then $\kappa(\Gamma u, ||\alpha||) \geq \pi_1 t$ by (d2). Hence, $|u|_{\pi_1 t'} \subseteq |u|_{\pi_1 t}$. As a consequence, $\kappa^*(P_r, |u|_{\pi_1 t'}) \leq \kappa^*(P_r, |u|_{\pi_1 t})$ by (A3). Hence, $\kappa^*(P_r, |u|_{\pi_1 t}) \geq \pi_2 t' \geq \pi_2 t$ by (d2), (d4). Thus, $u \in ||P_r||_t$ by the definition of graded meaning. In the sequel, $f_\$ ||P_r||_{t'} \subseteq f_\$ ||P_r||_t$ by (d1), (d3). Hence, $r \in \mathrm{apl}_{t'}^{f_\$}$ implies $r \in \mathrm{apl}_t^{f_\$}$ by the definition of graded applicability in the sense of $f_\$$. In case (f), for any rule r, $r \in \mathrm{Apl}_{t(1)} x$ iff $x \subseteq ||P_r||_t$ iff $\forall u \in x. u \in ||P_r||_t$ iff $\forall u \in x. r \in \mathrm{apl}_t u$ iff $r \in \bigcap \{\mathrm{apl}_t u \mid u \in x\}$. □

Let us briefly comment the results. By (a), rough applicability of a rule to u in the sense of pos_s and the graded applicability of a rule to Γu coincide. (b) is a direct consequence of the properties of approximation mappings. (c) states that the fundamental notion of rough applicability as well as the graded forms of applicability in the sense of $\mathrm{upp} \circ f_\$$ and $\mathrm{pos}_s \circ f_\$$ are determined up to granulation of information. By (d), if $t \preceq t'$, then every rule which is t'-applicable to an object u in the sense of a monotone mapping $f_\$$, is t-applicable to u in the very sense as well. It follows by (e) that the t-applicability with certainty implies the fundamental t-applicability, and the latter form implies the possible t-applicability. Finally, (f) gives a characterization of the $t(1)$-applicability of a rule to a set of objects x in terms of the fundamental t-applicability of the rule to elements of x.

Theorem 2. *Let u be any object, x, x' – sets of objects, r, r' – rules, $s'' \in T$, $s, s' \in T_1$ such that $\pi_1 s' \neq c$, and $t, t' \in T_2$. In cases (j), (k), assume also that κ is quasi-standard. Then, we have:*

(a) $r \in \mathrm{Apl}_{(s'',1,1)} x$ iff $P_r \subseteq |x|_{(s'',1)}$.

(b) $\mathrm{Apl}_{s(1)} U = \{r \in \mathrm{RUL} \mid ||P_r||_s = U\}$.

(c) $\mathrm{Apl}_{s(0)} x = \mathrm{Apl}_{(0)s'} x = \mathrm{Apl}_t \emptyset = \mathrm{RUL}$.

(d) If $\pi_3 t > 0$, then $\mathrm{Apl}_t\{u\} = \mathrm{apl}_{\lhd t} u$.

(e) If $t \preceq t'$, then $\mathrm{Apl}_{t'} \sqsubseteq \mathrm{Apl}_t$.

(f) $\mathrm{Apl}_{(1)s'} \sqsubseteq \mathrm{Apl}_{(s'')s'} \sqsubseteq \mathrm{Apl}_{(0)s'}$.

(g) $\bigcap_{x \subseteq U} \bigcap_{t \in T_2} \mathrm{Apl}_t x = \mathrm{Apl}_{(1)^3} U = \mathrm{Apl}_{(c,1,1)} U = \{r \in \mathrm{RUL} \mid ||P_r|| = U\}$.

(h) If $P_r \subseteq P_{r'}$ and $\pi_2 t = 1$, then $r' \in \mathrm{Apl}_t x$ implies $r \in \mathrm{Apl}_t x$.

(i) If $\exists \alpha \in P_r.||\alpha||_{\pi_1 t} = \emptyset$, $\pi_2 t = 1$, and $\pi_3 t > 0$, then $r \in \mathrm{Apl}_t x$ iff $x = \emptyset$.

(j) If $x' \cap ||P_r||_{\lhd t} = \emptyset$, then $r \in \mathrm{Apl}_t(x \cup x')$ implies $r \in \mathrm{Apl}_t x$ and $r \in \mathrm{Apl}_t x$ implies $r \in \mathrm{Apl}_t(x - x')$.

(k) If $x' \subseteq ||P_r||_{\lhd t}$ and $r \in \mathrm{Apl}_t(x - x')$, then $r \in \mathrm{Apl}_t x$.

Proof. We prove (g) only. First, note that (g1) $\mathrm{Apl}_{(1)^3} U = \{r \in \mathrm{RUL} \mid ||P_r||_{(1,1)} = U\}$ and (g2) $\mathrm{Apl}_{(c,1,1)} U = \{r \in \mathrm{RUL} \mid ||P_r|| = U\}$ by (b). It is easy to see that for any object u and a formula α, $u \models_1 \alpha$ implies $u \models \alpha$. Indeed, if $u \models_1 \alpha$, then $\Gamma u \subseteq ||\alpha||$. Since $u \in \Gamma u$, it holds $u \models \alpha$ as required. As a consequence, (g3) $|u|_1 \subseteq |u|$. In the next step, we prove (g4) $||P_r||_{(1,1)} = U$ iff $||P_r|| = U$ (recall that $||P_r|| = ||P_r||_{(c,1)}$). "$\Rightarrow$" Assume $||P_r||_{(1,1)} = U$. Hence, for every object u, $P_r \subseteq |u|_1$ by the definition of $(1,1)$-meaning. In virtue of (g3), $P_r \subseteq |u|$. Hence $||P_r|| = U$ by the definition of meaning. "$\Leftarrow$" Assume $||P_r|| = U$. Hence, for every object u, $P_r \subseteq |u|$ by the definition of meaning. In other words, $\forall u \in U.\forall \alpha \in P_r.u \in ||\alpha||$, i.e., $\forall \alpha \in P_r.||\alpha|| = U$. Hence, $\forall u \in U.\forall \alpha \in P_r.\Gamma u \subseteq ||\alpha||$. Thus, $\forall u \in U.\forall \alpha \in P_r.u \models_1 \alpha$ by the definition of $\models_1$, i.e., $\forall u \in U.\forall \alpha \in P_r.\alpha \in |u|_1$, i.e., $\forall u \in U.P_r \subseteq |u|_1$. Hence, $||P_r||_{(1,1)} = U$ by the definition of $(1,1)$-meaning. By (g1), (g2), and (g4), it holds that (g5) $\mathrm{Apl}_{(1)^3} U = \mathrm{Apl}_{(c,1,1)} U$. Observe that (g6) for any $x \subseteq U$, $\bigcap\{\mathrm{Apl}_t x \mid t \in T_2\} = \mathrm{Apl}_{(1)^3} x$ by (e), (f). Next, we show that (g7) $\bigcap\{\mathrm{Apl}_{(1)^3} x \mid x \subseteq U\} = \mathrm{Apl}_{(1)^3} U$. "$\subseteq$" is obvious. To prove "$\supseteq$", consider a rule $r \in \mathrm{Apl}_{(1)^3} U$. By the definition of $(1,1,1)$-applicability, $U \subseteq ||P_r||_{(1,1)}$. Hence, for any $x \subseteq U$, $x \subseteq ||P_r||_{(1,1)}$. Again by the definition of $(1,1,1)$-applicability, $r \in \mathrm{Apl}_{(1)^3} x$ for every set of objects x. Hence, $r \in \bigcap\{\mathrm{Apl}_{(1)^3} x \mid x \subseteq U\}$. Thus, $\bigcap\{\mathrm{Apl}_t x \mid x \subseteq U \wedge t \in T_2\} = \mathrm{Apl}_{(1)^3} U$ by (g6), (g7). Hence, (g) finally follows by (g2), (g5). □

Some comments can be handy. First, as directly follows from the definitions of applicability, the $(c,1,1)$-applicability is the same as the classical applicability. Next, if $\pi_2 t = \pi_3 t = 1$, then a rule r is t-applicable to a set of objects x iff every premise of r is $\lhd t$-satisfied for x by (a). If $\pi_3 t = 1$, then a rule r is t-applicable to the whole universe U iff U is the $\lhd t$-meaning of the set of premises of r in virtue of (b). By (c), every rule is t-applicable to the empty set as well as $s(0)$- and $(0)s'$-applicable to any sets of objects. If $\pi_3 t > 0$, then the t-applicability of a rule to $\{u\}$ is the same as the $\lhd t$-applicability of the rule to u by (d). Property (e) states that if t' is greater than or equal t in the sense of $\preceq$, then every rule t'-applicable to a set of objects x is t-applicable to x as well.[4] It follows from (e) and (f) that $\mathrm{Apl}_{(1)^3}$ and $\mathrm{Apl}_{(0)^3}$ are the least and the greatest elements of the partially ordered family of mappings Apl_t, respectively. By (g), the following sets of rules are identical: the set of all rules t-applicable to all sets of objects for each $t \in T_2$; the set of all rules $(1,1,1)$-applicable to U; the set of all rules $(c,1,1)$-applicable to U; and the set of all rules of which every premise is satisfied for each object of U. Hence, axiomatic rules (i.e., rules without premises) are t-applicable to every set of objects for each $t \in T_2$ since $||\emptyset|| = U$. In virtue of (h), if $\pi_2 t = 1$ and a rule r' is t-applicable to a set of objects x, then every rule of which premises are also premises of r' is t-applicable to x as well. By (i), if $\pi_2 t = 1$, $\pi_3 t > 0$, and some premise of a rule r is $\pi_1 t$-unsatisfiable, then r is t-applicable to the empty set only. Recall that RIFs are quasi-standard in cases (j), (k). (j) states that the property of being inapplicable (resp., applicable) in the sense of Apl_t is invariant under adding (removing) objects for which sets of premises of rules are $\lhd t$-unsatisfiable. Finally, (k) says that the property of being inapplicable in the sense of Apl_t is invariant under removing objects for which sets of premises of rules are $\lhd t$-satisfiable.

5 Summary

The aim of this paper was to further analyze rough applicability of rules. We generalized the fundamental concept of graded applicability in two ways, where, nevertheless, all premises of a rule were treated on equal terms. In the future, rules with premises partitioned into classes will be of interest. Applicability is only one aspect of application of rules. An analysis of the results of rough application and the question of rough quality of rules are of importance as well. The latter problem is closely related to propagation of uncertainty. Obviously, not all concepts of rough applicability can prove useful from the practical point of view. Nevertheless, some of them deserve our attention as they seem to describe formally certain forms of soft applicability of rules, observed in real life situations.

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[4] In other words, if viewed as a mapping of two variables, the graded applicability of a rule to a set of objects Apl is co-monotone in the variable t.

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