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A Graded Meaning of Formulas in Approximation Spaces

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Abstract. The aim of the paper is to introduce degrees of satisfiability as well as a graded form of the meaning of formulas and their sets in the approximation space framework.

Keywords: granulation of information, Pawlak information systems, approximation space, graded meaning of formulas

To Alberto and Maurizio

1. Introduction

Leaving aside philosophical disputes on what, actually, the *meaning* of a formula is and how to represent it, we shall identify the meaning of a formula with a set of objects of some sort, having some properties.

In knowledge representation systems like Pawlak information systems [15, 16] and parameterized approximation spaces [24], an object is considered together with objects attached to it in the form of a granule of information. Granulation of information is caused, among others, by incompleteness of knowledge about objects, available to an observer of a phenomenon or to a user of an information system.

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The crisp notion of the meaning of formulas or sets of formulas, interpreted in such granulated structures, evolves in a natural way to a more fine-grained notion of the graded meaning, with degrees in the unit interval $[0,1]$. Chakraborty and Basu [1, 2, 4, 5] studied a general notion of a graded logical consequence (and a related notion of graded consistency of a set of formulas) in the fuzzy set framework. A rough consequence was introduced by Chakraborty and Banerjee in [3] (see also [4]). Up to our knowledge, however, no graded form of consequence was investigated in the generalized rough set setting. In this article, we try to fill up this gap by introducing and studying graded notions of satisfiability and meaning, as well as graded forms of entailment, consequence, and truth, relativized to a given approximation space. In future, these notions will be used to investigate graded application of rules, their meaning, and quality.

Throughout the paper, the cardinality of a set x, the power set of x, and the Cartesian product of x, taken $n > 1$ times, will be denoted by $\#x$, $\wp(x)$, and x^n, respectively. Let $k \geq 2$, $i = 1, \ldots, k$, $(x_i, \leq_i)$ be partially ordered sets, and s, t be k-tuples belonging to $x_1 \times \ldots \times x_k$. Then, $\pi_i(t)$ denotes the i-th element of t and $<_i$ – the strict version of $\leq_i$. By $\preceq$ and $\prec$ we denote the following partial ordering relations:

$$\begin{aligned} s \preceq t &\overset{\text{def}}{\leftrightarrow} \forall i = 1, \ldots, k.\pi_i(s) \leq_i \pi_i(t). \\ s \prec t &\overset{\text{def}}{\leftrightarrow} \forall i = 1, \ldots, k.\pi_i(s) <_i \pi_i(t). \end{aligned} \tag{1}$$

$\succeq$ and $\succ$ denote the converse relations of $\preceq$ and $\prec$, respectively. For any sets x, x_0, y, y_0 such that $x_0 \subseteq x$ and $y_0 \subseteq y$, and a relation $\varrho \subseteq x \times y$, we denote the image of x_0 by $\varrho^{\rightarrow}(x_0)$, whereas the inverse image of y_0 by $\varrho^{\leftarrow}(y_0)$.

Let $T \overset{\text{def}}{=} [0,1] \cup \{c\}$, where $[0,1]$ is the unit interval of reals and c is a constant to denote "crisp" as opposite to "vague". By assumption, $c \leq c$, where $\leq$ extends the natural ordering of reals to T. Additionally, let $T_1 \overset{\text{def}}{=} T \times [0,1]$. For simplicity, parentheses and "c" will be omitted in formulas if no confusion results.

In Sect. 2, we concisely recall the idea of granulation of information. Next, the concepts of rough inclusion and membership functions are reviewed. In Sect. 3, starting with the Pawlak information systems, we recall the notion of an approximation space. A graded meaning of a formula, and related notions are defined in Sect. 4. A generalization of the concept of graded meaning to the case of a set of formulas is presented in Sect. 5. Graded forms of entailment, consequence, and truth, relativized to a given approximation space, are proposed in Sect. 6. Section 7 contains a brief summary.

2. Granulation Mappings, Rough Inclusion Functions, and Rough Membership Functions

Consider a non-empty set U whose elements, denoted by u, v with subscripts whenever needed, can be thought of as objects. In accordance with [31, 12], a granule of information is a set of objects, drawn together and/or toward some object on the base of similarity. From this pespective, a *granulation mapping* on U is any mapping on U, assigning granules of information to objects of U. Throughout the paper, we shall mainly consider granulation mappings of the form $\Gamma : U \mapsto \wp U$, where $u \in \Gamma(u)$ for every $u \in U$. In words, Γ assigns to an object u, a set of objects $\Gamma(u)$ which are in some sense similar to u. It is also assumed that objects are similar to themselves. Granulation mappings of this kind are known as *uncertainty* mappings [24]. A constructive definition of an exemplary uncertainty mapping is given in [24, 14].

The idea underlying the notion of a *rough inclusion function* (RIF for short) goes back to Łukasiewicz [13]. For a given set of objects U, a RIF is a mapping $\kappa : (\wp U)^2 \mapsto [0,1]$ which assigns to every pair (x,y) of subsets of U, a number $0 \leq k \leq 1$ expressing the degree of inclusion of x in y. In the literature, several RIFs are considered [6, 18, 24, 29]. Polkowski and Skowron [18, 19, 21, 22] generalized *Mereology*, a formal theory of the relationship of being-a-part founded by Leśniewski [11], to *Rough Mereology*, a theory of being-a-part-in-a-degree, based on the formal notion of a RIF. The best known RIF, called *standard*, has a probabilistic flavor as it is based on the frequency count. Given a non-empty finite set U, the standard RIF is the mapping $\kappa^{st} : (\wp U)^2 \mapsto [0,1]$ such that for any $x, y \subseteq U$,

$$\kappa^{st}(x,y) \stackrel{\text{def}}{=} \begin{cases} \frac{\#(x\cap y)}{\#x} & \text{if } x \neq \emptyset \\ 1 & \text{otherwise.} \end{cases} \tag{2}$$

If arbitrary non-empty sets of objects are allowed, the above definition is still meaningful provided that the first argument is finite.

In our approach, any RIF $\kappa : (\wp U)^2 \mapsto [0,1]$ is supposed to satisfy the following conditions, for any sets of objects x, y, z:

(A1) $\kappa(x,y) = 1$ iff $x \subseteq y$.
(A2) If $x \neq \emptyset$, then $\kappa(x,y) = 0$ iff $x \cap y = \emptyset$.
(A3) If $y \subseteq z$, then $\kappa(x,y) \leq \kappa(x,z)$.

First of all, let us note that κ^{st} satisfies (A1)–(A3). Conditions (A1), (A3) are postulated, in a slightly different form, by Rough Mereology. (A2) may be viewed as too strong, limiting the class of RIFs to these resembling the standard RIF. However, a similar argument can be used in the case of (A1). For example, let us consider a mapping $f : (\wp U)^2 \mapsto [0,1]$ such that $f(x,y) = \kappa(\bigcup \Gamma^{\rightarrow}(x), \bigcup \Gamma^{\rightarrow}(y))$, where κ is a RIF. f measures the degree of inclusion of a set x in a set y by measuring the degree of inclusion of the granule $\bigcup \Gamma^{\rightarrow}(x)$, associated with x, in the granule $\bigcup \Gamma^{\rightarrow}(y)$, associated with y. Intuitively, f is a kind of rough inclusion mapping. On the other hand, it may not satisfy (A1). In future, it would be interesting to re-elaborate the problem of graded meaning, starting with some weaker versions of (A1)–(A3).

By a *quasi-standard* RIF we understand any RIF satisfying (A1)–(A3) and defined by (2) for finite first arguments.[1]

Proposition 2.1. For any quasi-standard RIF κ, finite sets of objects x, x_1, x_2, and any set of objects y, it holds that:

(a) If $x \neq \emptyset$, then $\kappa(x,y) + \kappa(x, U - y) = 1$.
(b) If $x_2 \cap y = \emptyset$, then $\kappa(x_1 \cup x_2, y) \leq \kappa(x_1, y) \leq \kappa(x_1 - x_2, y)$.
(c) If $x_2 \subseteq y$, then $\kappa(x_1 - x_2, y) \leq \kappa(x_1, y)$.

[1] Notice that the standard RIF is quasi-standard as well.

The proof is left as an exercise.

Apart from RIFs, it can be useful to consider *rough membership functions* (RMFs for short) [17, 20] which generalize the relationship of being-a-member of a set. For any $x \subseteq U$, by a *rough x-membership function* (x-RMF) we understand a function $\mu_x : U \mapsto [0, 1]$, measuring the degree of membership of elements of U in x. Clearly, $\mu_U(u) = 1$ and $\mu_\emptyset(u) = 0$. Where U is finite and $\Gamma^{\rightarrow}(U)$ is a partition of U, the *standard* x-RMF, μ_x^{st}, is defined by

$$\mu_x^{st}(u) \stackrel{\text{def}}{=} \frac{\#(\Gamma(u) \cap x)}{\#\Gamma(u)}, \text{ i.e., } \mu_x^{st}(u) = \kappa^{st}(\Gamma(u), x). \tag{3}$$

Observe that the definition is also meaningful for any non-empty U and any granulation mapping Γ, where $\Gamma(u)$ is finite for every u. In such cases, slightly abusing the terminology, the term "standard" will be used to RMFs defined by the first equality of (3). Moreover, the latter equality may be taken as the definition of a x-RMF μ_x in a general case, given an uncertainty mapping Γ and a RIF κ. Then, $u \in x$ implies $\mu_x(u) > 0$, whereas $u \notin x$ implies $\mu_x(u) < 1$.

3. Approximation Spaces

Recall that any pair (U, A) of non-empty finite sets of objects and attributes, respectively, is called a *Pawlak information system* [15, 16]. Each attribute $a \in A$ is a mapping $a : U \mapsto V_a$, assigning to every object u, a value $a(u) \in V_a$. Any subset B of A induces an equivalence relation (known as the *B-indiscernibility* relation) on U, ind_B, such that for any $u, v \in U$,

$$(u, v) \in \text{ind}_B \stackrel{\text{def}}{\leftrightarrow} \forall a \in B. a(u) = a(v). \tag{4}$$

The corresponding uncertainty mapping Γ_B is defined by $\Gamma_B(u) \stackrel{\text{def}}{=} \text{ind}_B^{\leftarrow}(\{u\})$. Notice that $\Gamma_B^{\rightarrow}(U)$ is a partition of U. The pair (U, Γ_B) forms a *rough approximation space*, where the *lower* and *upper rough approximations* of a set of objects x, $\text{low}_B(x)$ and $\text{upp}_B(x)$, respectively, are defined as follows:

$$\text{low}_B(x) \stackrel{\text{def}}{=} \bigcup\{\Gamma_B(u) \mid \Gamma_B(u) \subseteq x\} \text{ and } \text{upp}_B(x) \stackrel{\text{def}}{=} \bigcup\{\Gamma_B(u) \mid \Gamma_B(u) \cap x \neq \emptyset\}. \tag{5}$$

Observe that

$$\text{low}_B(x) = \{u \mid \Gamma_B(u) \subseteq x\} \text{ and } \text{upp}_B(x) = \{u \mid \Gamma_B(u) \cap x \neq \emptyset\}. \tag{6}$$

Let us recall that for any RIF κ satisfying (A1)–(A3) and any x-RMF μ_x defined by the second equality of (3),

$$\begin{aligned} &\Gamma_B(u) \subseteq x \text{ iff } \kappa(\Gamma_B(u), x) = 1 \text{ iff } \mu_x(u) = 1 \text{ and} \\ &\Gamma(u) \cap x \neq \emptyset \text{ iff } \kappa(\Gamma(u), x) > 0 \text{ iff } \mu_x(u) > 0. \end{aligned} \tag{7}$$

A first natural generalization of the above notion of AS is obtained by starting with an equivalence relation $\varrho \subseteq U^2$ and taking $\Gamma(u) \stackrel{\text{def}}{=} \varrho^{\leftarrow}(\{u\})$. In the next step, the conditions that U be finite and $\Gamma^{\rightarrow}(U)$ be a partition of U are relaxed. Skowron and Stepaniuk [24] proposed a general notion of a *parameterized approximation space* as a triple $\mathcal{A}_\$ = (U, \Gamma_\$, \kappa_\$)$, where U is a non-empty set of objects, $\Gamma_\$$ is an *uncertainty* mapping (i.e., $\Gamma_\$: U \mapsto \wp U$ and for each $u \in U$, $u \in \Gamma_\$(u)$), $\kappa_\$$ is a RIF satisfying the axioms of Rough Mereology, and \$ is a list of tuning parameters to achieve a satisfactory quality of approximation. As earlier, one may alternatively start with a reflexive relation $\varrho_\$ \subseteq U^2$ and take $\Gamma_\$(u) \stackrel{\mathrm{def}}{=} \varrho_\$^{\leftarrow}(\{u\})$. Let us note that $\Gamma_\$^{\rightarrow}(U)$ is a covering of U. Granules of the form $\Gamma_\$(u)$ are viewed as *elementary*. Set-theoretical unions of elementary granules of information form definable sets of objects. More formally, $x \subseteq U$ is *definable* if there is a set $y \subseteq U$ such that $x = \bigcup \Gamma_\$^{\rightarrow}(y)$. In $\mathcal{A}_\$$, any set of objects (concept) x may be approximated by its *lower* and *upper rough approximations*, $\mathrm{low}_\$(x)$ and $\mathrm{upp}_\$(x)$, respectively, defined as follows:

$$\mathrm{low}_\$(x) \stackrel{\mathrm{def}}{=} \{u \mid \kappa_\$(\Gamma_\$(u), x) = 1\} \text{ and } \mathrm{upp}_\$(x) \stackrel{\mathrm{def}}{=} \{u \mid \kappa_\$(\Gamma_\$(u), x) > 0\}. \tag{8}$$

Henceforth, we shall omit the parameters \$ for simplicity. In our approach, an *approximation space* is a triple $\mathcal{A} = (U, \Gamma, \kappa)$, where U and Γ are as earlier and κ is a RIF satisfying (A1)–(A3). In ASs, sets of objects (concepts) may be approximated in terms of lower and upper rough approximations, defined in varied ways, or by means of variable-precision positive and negative regions [7, 9, 10, 23, 24, 25, 26, 28, 27, 30, 32, 33].

4. The Meaning of Formulas

Consider an AS $\mathcal{A}$ as earlier and a formal language L to express properties of $\mathcal{A}$. Formulas of L are denoted by the lowercase Greek letters α, β, γ with subscripts if needed. All formulas of L constitute a set FOR. Assume that commutative conjunction ($\wedge$) and disjunction ($\vee$) occur among the connectives of L. Then, for any non-empty finite set of formulas X, $\bigwedge X$ and $\bigvee X$ denote a conjunction and a disjunction of all elements of X, respectively.

Starting with a relation of *satisfiability* of formulas for objects of U, $\models_c$, the (crisp) *meaning* (or c-*meaning*) of α, written $||\alpha||_c$, is defined as the extension of α, i.e., the set of all objects of U that α is satisfied for:

$$||\alpha||_c \stackrel{\mathrm{def}}{=} \{u \mid u \models_c \alpha\}. \tag{9}$$

$u \models_c \alpha$ reads as "α is c-*satisfied* (or simply *satisfied*) for u".[2]

Example 4.1. Consider an information system (U, A), where $U = \{2, \ldots, m\}$ and $A = \{a_1, \ldots, a_n\}$ for some natural numbers m, n. For simplicity, we use numbers to denote objects. A simple logical language is defined as follows. Individual variables over U are denoted by u, v with subscripts whenever needed. The only primitive predicate symbol is the binary symbol $=$, interpreted as equality. The only function symbols of non-zero arity are the unary function symbols $a_1, \ldots, a_n$. We use b, with subscripts if needed, to denote elements of $\bigcup_{a \in A} V_a$. Constant symbols are elements of U and $\bigcup_{a \in A} V_a$. Terms are individual variables, constant symbols, or sequences of symbols of the forms $a_i(u)$ and $a_i(j)$. Atomic formulas have the forms $a_i(u) = b$ or $a_i(j) = b$. Primitive propositional connectives are conjunction ($\wedge$) and negation ($\neg$). Formulas are atomic formulas and expressions, formed from them along the standard lines, using $\wedge$ and $\neg$. The remaining connectives, i.e., disjunction ($\vee$), implication ($\rightarrow$), and double implication ($\leftrightarrow$) are defined in the classical way in terms of $\wedge$ and $\neg$. $a_i(u) \neq b$ and $a_i(j) \neq b$ are

[2]We also say that α *is true* of or *holds* for u.

abbreviations for $\neg(a_i(u) = b)$ and $\neg(a_i(j) = b)$. For any formulas $a_i(u) = b, a_i(k) = b, \alpha, \beta$, and $j \in U$, the crisp satisfiability of formulas is defined as follows:

$$\begin{array}{ll} & j \models a_i(u) = b \text{ iff } a_i(j) = b. \\ & j \models a_i(k) = b \text{ iff } j = k \text{ and } a_i(k) = b. \\ & j \models \alpha \wedge \beta \text{ iff } j \models \alpha \text{ and } j \models \beta. \\ & j \models \neg\alpha \text{ iff } j \not\models \alpha. \\ \text{Hence,} & ||a_i(u) = b|| = \{j \in U \mid a_i(j) = b\}; \\ & ||a_i(j) = b|| = \begin{cases} \{j\} & \text{if } a_i(j) = b \\ \emptyset & \text{otherwise;} \end{cases} \\ & ||\alpha \wedge \beta|| = ||\alpha|| \cap ||\beta||; \\ & ||\neg\alpha|| = U - ||\alpha||. \end{array} \tag{10}$$

Single objects are perceived from the perspective of granules of information attached to them. Therefore, starting with crisp satisfiability of formulas and taking into account granulation of information, we arrive at the notion of graded satisfiability of formulas for objects.[3] For any $x \subseteq U$, let μ_x be the RMF defined by the second equality of (3). We say that α is *t-satisfied* for u,[4] $u \models_t \alpha$, if the degree of the membership of u in $||\alpha||$ is equal or greater than t. Formally,

$$u \models_t \alpha \overset{\text{def}}{\leftrightarrow} \mu_{||\alpha||}(u) \geq t, \text{ i.e., iff } \kappa(\Gamma(u), ||\alpha||) \geq t. \tag{11}$$

By the *t-meaning* of α, $||\alpha||_t$, we understand the set of all objects for which α is t-satisfied, i.e.,

$$||\alpha||_t \overset{\text{def}}{=} \{u \mid u \models_t \alpha\}. \tag{12}$$

Recalling that $T = [0, 1] \cup \{c\}$, let us present some properties of the notions introduced above.

Proposition 4.1. For any formulas α, β, a non-empty finite set of formulas X, $u, v \in U$, $s \in [0, 1]$, $t, t_1, t_2 \in T$, and assuming that κ is quasi-standard in (j), we have:

$$\begin{array}{ll} (a) & \text{If } \Gamma(u) = \Gamma(v), \text{ then } u \models_s \alpha \text{ iff } v \models_s \alpha. \\ (b) & \text{If } ||\alpha|| = \emptyset, \text{ then } ||\alpha||_t = \begin{cases} U & \text{if } t = 0 \\ \emptyset & \text{otherwise.} \end{cases} \\ (c) & \text{If } ||\alpha|| \subseteq ||\beta||, \text{ then } ||\alpha||_t \subseteq ||\beta||_t. \\ (d) & ||\bigwedge X||_t \subseteq \bigcap_{\alpha \in X} ||\alpha||_t. \\ (e) & ||\alpha \wedge \beta||_1 = ||\alpha||_1 \cap ||\beta||_1. \\ (f) & \bigcup_{\alpha \in X} ||\alpha||_t \subseteq ||\bigvee X||_t. \end{array}$$

[3]Since the idea is very natural, the definition may seem to be familiar. Up to our knowledge, its extension to the graded satisfiability of sets of formulas, as well as other related notions are new. For the sake of simplicity, we shall use the same symbol $\models_t$ in several contexts if no confusion results.

[4]In other words, α is *true* of or *holds* for u *in degree* t.

(g) If $t_1 \leq t_2$, then $||\alpha||_{t_2} \subseteq ||\alpha||_{t_1}$.

(h) $||\alpha||_1 \subseteq ||\alpha||_t \subseteq ||\alpha||_0 = U$.

(i) $||\alpha|| = U$ iff $||\alpha||_1 = U$ iff $\forall s \in [0,1].||\alpha||_s = U$.

(j) $(||\alpha||_1 \cap ||\beta||_t) \cup (||\alpha||_t \cap ||\beta||_1) \subseteq ||\alpha \wedge \beta||_t$.

Proof:
We only prove (c). Consider the non-trivial case, where $t \neq c$. Assume that $||\alpha|| \subseteq ||\beta||$ and $u \in ||\alpha||_t$. By definition, $\kappa(\Gamma(u), ||\alpha||) \geq t$. By assumption and (A3), $\kappa(\Gamma(u), ||\beta||) \geq t$. Thus, $u \in ||\beta||_t$. □

Let us note that with every formula α, there is associated a family $\{||\alpha||_t\}_{t \in T}$ of t-meanings of α, partially ordered by $\subseteq$. By the above proposition, $||\alpha||_1$ is the least element and $||\alpha||_0$ is the greatest one.

Example 4.2. Table 1 shows a fragment of an information system (U, A), values of Γ, and values of the standard $\mu_{||\alpha||}$, $\mu_{||\beta||}$, and $\mu_{||\gamma||}$. In this case, $U = \{2, \ldots, 12\}$, $a_1, a_2, a_3 \in A$, $b_1, \ldots, b_4 \in \bigcup_{a \in A} V_a$, and $*$ denotes some other values of attributes. Let α be $(a_1(u) = b_1 \wedge a_2(u) = b_2) \vee a_3(u) = b_4$, β be $a_2(u) \neq b_2 \wedge a_2(u) \neq b_3$, and γ be $\neg\alpha \vee \beta$. It is easy to see that $||\alpha|| = \{2, 4, 6, 8, 11, 12\}$, $||\beta|| = \{4, 5, 11\}$, and $||\gamma|| = \{3, 4, 5, 7, 9, 10, 11\}$. The t-meaning of α, β, γ ($t > 0$) is given in Table 2. Recall that $||\alpha||_0 = ||\beta||_0 = ||\gamma||_0 = U$.

Table 1.

u	a_1	a_2	a_3	$\Gamma(u)$	$\mu_{\|\|\alpha\|\|}(u)$	$\mu_{\|\|\beta\|\|}(u)$	$\mu_{\|\|\gamma\|\|}(u)$
2	b_1	b_2	b_4	$\{2,6\}$	1	0	0
3	b_1	b_3	$*$	$\{3,5,9\}$	0	$\frac{1}{3}$	1
4	$*$	$*$	b_4	$\{4,11\}$	1	1	1
5	$*$	$*$	$*$	$\{4,5\}$	$\frac{1}{2}$	1	1
6	b_1	b_2	$*$	$\{2,6,12\}$	1	0	0
7	$*$	b_2	$*$	$\{4,7,8\}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
8	b_1	b_3	b_4	$\{3,8\}$	$\frac{1}{2}$	0	$\frac{1}{2}$
9	$*$	b_3	$*$	$\{9,10\}$	0	0	1
10	$*$	b_3	$*$	$\{3,9,10\}$	0	0	1
11	$*$	$*$	b_4	$\{2,11\}$	1	$\frac{1}{2}$	$\frac{1}{2}$
12	b_1	b_2	$*$	$\{6,12\}$	1	0	0

Given $t \in T$, the set of all formulas which are t-satisfied for u is denoted by $|u|_t$:

$$|u|_t \stackrel{\text{def}}{=} \{\alpha \mid u \models_t \alpha\}. \tag{13}$$

Recall that $T_1 = T \times [0,1]$. The relation of graded satisfiability of formulas for objects may be generalized on the left-hand side to a relation of t-satisfiability of formulas for sets of objects, where $t = (t_1, t_2) \in T_1$. For any set of objects x and a formula α, let

$$x \models_t \alpha \stackrel{\text{def}}{\Leftrightarrow} \kappa(x, ||\alpha||_{t_1}) \geq t_2 \text{ and } |x|_t \stackrel{\text{def}}{=} \{\alpha \mid x \models_t \alpha\}. \tag{14}$$

Table 2.

t	$(0, \frac{1}{3}]$	$(\frac{1}{3}, \frac{1}{2}]$	$(\frac{1}{2}, \frac{2}{3}]$	$(\frac{2}{3}, 1]$
$\|\|\alpha\|\|_t$	$U - \{3, 9, 10\}$	$U - \{3, 7, 9, 10\}$	$\{2, 4, 6, 11, 12\}$	$\{2, 4, 6, 11, 12\}$
$\|\|\beta\|\|_t$	$\{3, 4, 5, 7, 11\}$	$\{4, 5, 11\}$	$\{4, 5\}$	$\{4, 5\}$
$\|\|\gamma\|\|_t$	$U - \{2, 6, 12\}$	$U - \{2, 6, 12\}$	$\{3, 4, 5, 7, 9, 10\}$	$\{3, 4, 5, 9, 10\}$

$x \models_t \alpha$ reads as "α is t-*satisfied* for x".[5] The underlying idea is that α is t-satisfied for x if for sufficiently many elements of x, α is satisfied in a sufficient degree, where sufficiency is determined by t.

Proposition 4.2. For any objects u, v, sets of objects x, y, a family $z \subseteq \wp U$, a formula α, $s_0 \in [0, 1]$, $s, s_1, s_2 \in T$, and $t, t_1, t_2 \in T_1$, we have:

$$
\begin{array}{ll}
(a) & \text{If } \Gamma(u) = \Gamma(v), \text{ then } |u|_{s_0} = |v|_{s_0}. \\
(b) & \alpha \in |u|_s \text{ iff } u \in ||\alpha||_s. \\
(c) & \alpha \in |x|_{(s,1)} \text{ iff } x \subseteq ||\alpha||_s, \text{ and } |x|_{(s,1)} = \bigcap_{u \in x} |u|_s. \\
(d) & |\{u\}|_t = \begin{cases} |u|_{\pi_1(t)} & \text{if } \pi_2(t) > 0 \\ \text{FOR} & \text{otherwise.} \end{cases} \\
(e) & |\emptyset|_t = \text{FOR}. \\
(f) & |\Gamma(u)|_{(c,s_0)} = |u|_{s_0}. \\
(g) & \text{If } s_1 \leq s_2, \text{ then } |u|_{s_2} \subseteq |u|_{s_1}. \\
(h) & |u|_1 \subseteq |u|_s \subseteq |u|_0 = \text{FOR}. \\
(i) & \text{If } t_1 \preceq t_2, \text{ then } |x|_{t_2} \subseteq |x|_{t_1}. \\
(j) & \text{If } x \subseteq y, \text{ then } |y|_{(s,1)} \subseteq |x|_{(s,1)}. \\
(k) & |\bigcup z|_{(s,1)} = \bigcap_{u \in \bigcup z} |u|_s = \bigcap_{x \in z} |x|_{(s,1)}.
\end{array}
$$

Proof:
We only show (i), leaving the rest as an exercise. Assume $t_1 \preceq t_2$ and consider the non-trivial case, where $\pi_1(t_1), \pi_1(t_2) \neq c$. Suppose that $\alpha \in |x|_{t_2}$. By definition, $\kappa(x, ||\alpha||_{\pi_1(t_2)}) \geq \pi_2(t_2)$. By assumption, Proposition 4.1(g), and (A3), $\kappa(x, ||\alpha||_{\pi_1(t_1)}) \geq \kappa(x, ||\alpha||_{\pi_1(t_2)}) \geq \pi_2(t_2) \geq \pi_2(t_1)$. Hence, $\alpha \in |x|_{t_1}$ by definition. □

Observe, in particular, that $\alpha \in |U|_{(s,1)}$ iff $||\alpha||_s = U$ in virtue of (c). Moreover, if $z \neq \emptyset$, then $|\bigcup z|_{(s,1)} \subseteq |\bigcap z|_{(s,1)}$ by (j).

Example 4.3. (Continuation.) Let $x = \{2, 3, 4\}$ and $t = (t_1, t_2) \in T_1$. Then, $\alpha \in |x|_t$ iff $t_1 = 0$ or $t_2 \leq \frac{2}{3}$, and the same for γ. Next, $\beta \in |x|_t$ iff $t_1 = 0$ or $(t_1 = c \vee t_1 > \frac{1}{3}) \wedge t_2 \leq \frac{1}{3}$ or $t_1 \leq \frac{1}{3} \wedge t_2 \leq \frac{2}{3}$.

[5] Equivalently, α is *true* of or *holds* for x *in degree* t.

5. The Meaning of Sets of Formulas

Along the standard lines, the relation of crisp satisfiability of formulas is extended on the right-hand side to the case of sets of formulas. Thus, for any object u and a set of formulas X,

$$\begin{aligned} u \models_c X &\overset{\text{def}}{\leftrightarrow} \forall \alpha \in X. u \models_c \alpha; \\ ||X||_c &\overset{\text{def}}{=} \{u \mid u \models_c X\}, \text{ i.e., } ||X||_c = \bigcap_{\alpha \in X} ||\alpha||. \end{aligned} \tag{15}$$

Consider a RIF $\kappa^* : (\wp\text{FOR})^2 \mapsto [0,1]$. For any $t = (t_1, t_2) \in T_1$, the t-satisfiability of a set of formulas X for an object u and the *t-meaning* of X, $||X||_t$, are defined as follows:

$$u \models_t X \overset{\text{def}}{\leftrightarrow} \kappa^*(X, |u|_{t_1}) \geq t_2 \text{ and } ||X||_t \overset{\text{def}}{=} \{u \mid u \models_t X\}. \tag{16}$$

$u \models_t X$ reads as "X is *t-satisfied* for u".[6] Informally speaking, X is t-satisfied for u if sufficiently many elements of X are satisfied for u in a sufficient degree, where sufficiency is determined by t.

Basic properties of the graded meaning of a set of formulas are presented below.

Proposition 5.1. For any objects u, v, a set of objects x, a formula α, sets of formulas X, Y, finite sets of formulas X_1, X_2, a family $Z \subseteq \wp\text{FOR}$, $s \in [0,1]$, $t \in T$, $t_1, t_2 \in T_1$, and assuming that κ^* is quasi-standard in (k), we have:

(a) If $\Gamma(u) = \Gamma(v)$, then $u \models_{t_1} X$ iff $v \models_{t_1} X$.

(b) $u \in ||X||_{(t,1)}$ iff $X \subseteq |u|_t$.

(c) $x \subseteq ||X||_{(t,1)}$ iff $X \subseteq |x|_{(t,1)}$.

(d) $||\emptyset||_{t_1} = U$ and $||\{\alpha\}||_{t_1} = \begin{cases} ||\alpha||_{\pi_1(t_1)} & \text{if } \pi_2(t_1) > 0 \\ U & \text{otherwise.} \end{cases}$

(e) $\bigcap_{\alpha \in X} ||\alpha||_t = ||X||_{(t,1)}$ and $\bigcup_{\alpha \in X} ||\alpha||_t = \bigcup_{s \in (0,1]} ||X||_{(t,s)}$.

(f) If $t_1 \preceq t_2$, then $||X||_{t_2} \subseteq ||X||_{t_1}$.

(g) $||X||_{(1,s)} \subseteq ||X||_{(t,s)} \subseteq ||X||_{(0,s)} = U$.

(h) $||X||_{(1,1)} = U$ iff $||X||_{(c,1)} = U$ iff $\forall t \in T_1. ||X||_t = U$.

(i) If $X \subseteq Y$, then $||Y||_{(t,1)} \subseteq ||X||_{(t,1)}$.

(j) If $\exists \alpha \in X. ||\alpha||_t = \emptyset$, then $||X||_{(t,1)} = \emptyset$.

(k) If $||X_2||_{(\pi_1(t_1),1)} = U$, then $||X_1 - X_2||_{t_1} \subseteq ||X_1||_{t_1}$.

(l) If $X_1 \neq \emptyset$, then $||\bigwedge X_1||_t \subseteq ||X_1||_{(t,1)}$.

(m) $||\bigcup Z||_{(t,1)} = \bigcap_{\alpha \in \bigcup Z} ||\alpha||_t = \bigcap_{X \in Z} ||X||_{(t,1)}$.

(n) If $||X \cap Y||_{(t,1)} = U$, then $||X - Y||_{(t,1)} = ||X||_{(t,1)}$.

(o) If $||Y - X||_{(t,1)} = U$, then $||X \cup Y||_{(t,1)} = ||X||_{(t,1)}$.

[6]In other words, X is *true* of or *holds* for u *in degree* t.

Proof:
We only prove (k). Assume $||X_2||_{(\pi_1(t_1),1)} = U$. Hence for every $u \in U$, $X_2 \subseteq |u|_{\pi_1(t_1)}$. Let $u \in ||X_1 - X_2||_{t_1}$. By definition, $\kappa^*(X_1 - X_2, |u|_{\pi_1(t_1)}) \geq \pi_2(t_1)$. In virtue of Proposition 2.1(c) and quasi-standardness of κ^*, $\kappa^*(X_1 - X_2, |u|_{\pi_1(t_1)}) \leq \kappa^*(X_1, |u|_{\pi_1(t_1)})$. In summary, $u \in ||X_1||_{t_1}$. □

There are striking similarities between Proposition 5.1 and Proposition 4.2. Namely, formulas correspond to objects and the t-meaning of a set of formulas $||X||_t$, where $t \in T_1$ is like the set $|x|_t$, where x is a set of objects. Next, observe that $||X|| = ||X||_{(c,1)}$ by (b). Another consequence of (b) is that $\alpha \in X$ implies $||X||_{(t,1)} \subseteq ||\alpha||_t$. Property (d) provides us, among others, with a criterion for replacement of a singleton by the formula constituting it. Broadly speaking, (f) and (i) express co-monotonicity of the mapping of graded meaning in both variables. Properties (k), (n), and (o) tell us which formulas of a given set are of minor importance when computing its graded meaning. In particular, $||\alpha||_t = U$ implies $||X - \{\alpha\}||_{(t,1)} = ||X \cup \{\alpha\}||_{(t,1)} = ||X||_{(t,1)}$ for any $t \in T$. Moreover, if X is finite, κ^* is quasi-standard, and $||\alpha||_{\pi_1(t_1)} = U$, it holds that $||X - \{\alpha\}||_{t_1} \subseteq ||X||_{t_1}$. Since the inclusion cannot be reversed in (1), a finite set of formulas cannot be replaced by a conjunction of its elements in a general case. In connection with (m), observe that if $Z \neq \emptyset$, then $||\bigcup Z||_{(t,1)} \subseteq ||\bigcap Z||_{(t,1)}$ by (i).

Example 5.1. (Continuation.) Let $X = \{\alpha, \gamma\}$ (i.e., X is a set of premises of *modus ponens*), κ^* be quasi-standard, and $t = (t_1, t_2)$. The t-meaning of X is given in Table 3.

Table 3.

$t_2 \backslash t_1$	0	$(0, \frac{1}{3}]$	$(\frac{1}{3}, \frac{1}{2}]$	$(\frac{1}{2}, \frac{2}{3}]$	$(\frac{2}{3}, 1]$	c
0	U	U	U	U	U	U
$(0, \frac{1}{2}]$	U	U	U	$U - \{8\}$	$U - \{7, 8\}$	U
$(\frac{1}{2}, 1]$	U	$\{4, 5, 7, 8, 11\}$	$\{4, 5, 8, 11\}$	$\{4\}$	$\{4\}$	$\{4, 11\}$

6. A Graded Form of Entailment

In this section, we introduce and investigate graded forms of entailment, consequence, and truth, relativized to an AS $\mathcal{A}$ as earlier. First, a set of formulas X is said to *entail* a set of formulas Y in a crisp sense,[7] $X \models_c Y$, iff satisfaction of X implies satisfaction of Y, for every u. Formally,

$$X \models_c Y \overset{\text{def}}{\leftrightarrow} \forall u.(u \models X \Rightarrow u \models Y), \text{ i.e., iff } ||X|| \subseteq ||Y||. \tag{17}$$

Consider $t = (t_1, t_2, t_3)$, $t_1, t_2 \in T_1$, and $t_3 \in [0, 1]$. For any sets of formulas X, Y, let

$$X \models_t Y \overset{\text{def}}{\leftrightarrow} \kappa(||X||_{t_1}, ||Y||_{t_2}) \geq t_3. \tag{18}$$

$X \models_t Y$, read as "Y is *t-entailed* by X", holds if for sufficiently many objects, a sufficient degree of satisfaction of X implies a sufficient degree of satisfaction of Y, where sufficiency is determined by t.

[7]Equivalently, X *c-entails* Y.

When $t_3 = 1$, $X \models_t Y$ iff $||X||_{t_1} \subseteq ||Y||_{t_2}$. Additionally, if $t_1 = t_2 = (c, 1)$, then $X \models_t Y$ iff $||X|| \subseteq ||Y||$ iff $X \models Y$. For simplicity, we shall write $X \models_t \beta$ and $\alpha \models_t \beta$ instead of $X \models_t \{\beta\}$ and $\{\alpha\} \models_t \{\beta\}$, respectively. Moreover, "$\emptyset$" will be omitted in expressions like $\emptyset \models_t Y$ and $\emptyset \models_t \beta$.[8] A formula α is said to be a *consequence* of X *in degree* t if $X \models_t \alpha$. When $\models_t \alpha$, we say that α is *true in degree* t. This means that for sufficiently many objects, α is satisfied in a sufficient degree, where sufficiency is determined by t.

Proposition 6.1. For any formulas α, β, sets of formulas X, Y, Z, $s = (s_1, s_2, s_3)$, $t = (t_1, t_2, t_3)$, where $s_i, t_i \in T_1$, $i = 1, 2$, and $s_3, t_3 \in [0, 1]$, it holds:

(a) If $||\alpha|| = U,$ then $X \models_t \alpha.$

(b) If $||\alpha|| = \emptyset$ and $\pi_1(t_2) = 0 \vee \pi_2(t_2) = 0 \vee t_3 = 0,$ then $X \models_t \alpha.$

(c) If $\alpha \in X$ and $\pi_2(t_2) = 0 \vee (\pi_2(t_1) = 1 \wedge \pi_1(t_2) \leq \pi_1(t_1)),$ then $X \models_t \alpha.$

(d) If $\alpha \models \beta$ and $\pi_2(t_2) = 0 \vee (\pi_2(t_1) > 0 \wedge \pi_1(t_2) \leq \pi_1(t_1)),$ then $\alpha \models_t \beta.$

(e) If $X \models_t \alpha$ and $\alpha \models \beta,$ then $X \models_t \beta.$

(f) If $\alpha \models_t \beta$ and $\pi_2(t_1) = 0,$ then $\models_t \alpha \to \beta.$

(g) $\models_t \emptyset.$

(h) If $t_2 \preceq t_1,$ then $X \models_t X.$

(i) If $X \models_t Y,$ $Z \subseteq Y,$ and $\pi_2(t_2) = 1,$ then $X \models_t Z.$

(j) If $X \models_t Y,$ $X \subseteq Z,$ and $\pi_2(t_1) = t_3 = 1,$ then $Z \models_t Y.$

(k) If $\pi_2(t_2) = t_3 = 1,$ then $X \models_t Y$ iff $\forall \alpha \in Y. X \models_t \alpha.$

(l) If $X \models_t Y,$ $s \preceq t,$ and $s_1 = t_1,$ then $X \models_s Y.$

(m) If $X \cup Z \models_t Y,$ $||Z - X||_{t_1} = U,$ and $\pi_2(t_1) = 1,$ then $X \models_t Y.$

Proof:
We only show (c) and (i). For (c) assume that (a1) $\alpha \in X$ and ($\pi_2(t_2) = 0$ or ((a2) $\pi_2(t_1) = 1$ and (a3) $\pi_1(t_2) \leq \pi_1(t_1)$)). If $\pi_2(t_2) = 0$, then $||\{\alpha\}||_{t_2} = U$ by Proposition 5.1(d). Hence, $\kappa(||X||_{t_1}, ||\{\alpha\}||_{t_2}) = 1 \geq t_3$. Assume to the contrary that $\pi_2(t_2) > 0$. By Proposition 5.1(d), $||\{\alpha\}||_{t_2} = ||\alpha||_{\pi_1(t_2)}$. By (a3) and Proposition 4.1(g), $||\alpha||_{\pi_1(t_1)} \subseteq ||\alpha||_{\pi_1(t_2)}$. By (a1), (a2), and Proposition 5.1(e), $||X||_{t_1} = \bigcap_{\beta \in X} ||\beta||_{\pi_1(t_1)} \subseteq ||\alpha||_{\pi_1(t_1)}$. Thus, $||X||_{t_1} \subseteq ||\alpha||_{\pi_1(t_2)}$. In summary, $\kappa(||X||_{t_1}, ||\{\alpha\}||_{t_2}) = 1 \geq t_3$. Finally, $X \models_t \alpha$ by definition. For (i) assume (b1) $X \models_t Y$, (b2) $Z \subseteq Y$, and (b3) $\pi_2(t_2) = 1$. $||Y||_{t_2} \subseteq ||Z||_{t_2}$ by (b2), (b3) and Proposition 5.1(i). Hence $t_3 \leq \kappa(||X||_{t_1}, ||Y||_{t_2}) \leq \kappa(||X||_{t_1}, ||Z||_{t_2})$ by (b1) and (A3). Thus, $X \models_t Z$. □

Let us note that in virtue of (d), $\alpha \models_t \alpha$ if $\pi_2(t_2) = 0$ or ($\pi_2(t_1) > 0$ and $\pi_1(t_2) \leq \pi_1(t_1)$).

There arises a question how our graded notions of consequence and truth are related to the concepts of graded consequence, rough consequence, and rough truth, defined by Chakraborty et al. [1, 2, 3, 4, 5]. For simplicity, the same symbols will be used to denote formulas in both formalisms in spite of different languages. Chakraborty's *graded consequence*, $\vdash^g$, is a fuzzy relation, relating sets of formulas to single formulas in the fuzzy set framework. The degree in which α is a consequence of X, written gr$(X \vdash^g \alpha)$,

[8] This resolves the problem of ambiguity in notation (cf. (14)).

is a member of the universe M of a complete lattice $(M, \leq, 0, 1)$. By definition, $\vdash^g$ satisfies the graded versions of the well-known axioms of the syntactical consequence relation:

$$\begin{array}{ll} (\mathrm{CH1}) & \text{If } \alpha \in X, \text{ then } \mathrm{gr}(X \vdash^g \alpha) = 1. \\ (\mathrm{CH2}) & \text{If } X \subseteq Z, \text{ then } \mathrm{gr}(X \vdash^g \alpha) \leq \mathrm{gr}(Z \vdash^g \alpha). \\ (\mathrm{CH3}) & \mathrm{gr}(X \cup Z \vdash^g \alpha) \wedge \inf_{\beta \in Z} \mathrm{gr}(X \vdash^g \beta) \leq \mathrm{gr}(X \vdash^g \alpha). \end{array}$$

The corresponding properties of our graded form of consequence, i.e., Proposition 6.1(c), (j), and (m) for $Y = \{\alpha\}$, are weaker.

The notion of a *rough consequence*, introduced by Chakraborty and Banerjee [3][9] in the classical rough set framework, refers to the concept of the upper rough approximation, defined by (5). According to their definition, a formula α is *roughly true* in a rough AS (U, Γ)[10], $|\!\approx \alpha$, iff the upper rough approximation of the meaning of α is the whole universe U. Formally,

$$|\!\approx \alpha \overset{\mathrm{def}}{\leftrightarrow} \mathrm{upp}||\alpha|| = U, \tag{19}$$

where upp is defined by (5). In our terms,

$$|\!\approx \alpha \text{ iff } ||\neg\alpha||_1 = \emptyset. \tag{20}$$

As a matter of fact, $\mathrm{upp}||\alpha|| = \{u \mid \Gamma(u) \cap ||\alpha|| \neq \emptyset\} = U - \{u \mid \Gamma(u) \subseteq U - ||\alpha||\} = U - \{u \mid \Gamma(u) \subseteq ||\neg\alpha||\} = U - ||\neg\alpha||_1 = U$ iff $||\neg\alpha||_1 = \emptyset$. Next, α is called a *rough consequence* of X, $X |\!\approx \alpha$, if the fact that all formulas of X are roughly true implies that α is true in the very sense. In formal terms,

$$X |\!\approx \alpha \overset{\mathrm{def}}{\leftrightarrow} \forall \beta \in X. |\!\approx \beta \Rightarrow |\!\approx \alpha. \tag{21}$$

Clearly, $|\!\approx \alpha$ iff $\emptyset |\!\approx \alpha$. Moreover, $X |\!\approx \alpha$ iff the fact that $X \subseteq \{\beta \mid ||\neg\beta||_1 = \emptyset\}$ implies $||\neg\alpha||_1 = \emptyset$.

7. Summary

Our main objective was to introduce and investigate graded forms of satisfiability and meaning of formulas and their sets in approximation spaces. Additionally, we have worked out the corresponding concepts of graded entailment, consequence, and truth, all relativized to a given AS. Moreover, relationships between ours and Chakraborty's approach were briefly discussed. In further research, the aim will be at application of the obtained concepts to study rules as well as their sets and complexes in semantical terms.

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[9] See also [4].

[10] Notice that κ is useless here and, hence, it may be omitted.

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