Topos and Quasitopos of Rough Sets

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The category theoretic properties of rough sets were first given by Banerjee and Chakraborty [1], and the category *ROUGH* was proposed. Later, Li and Yuan [2] defined another category *RSC* of rough sets. In this work, we observe that *RSC* is, in fact, equivalent to *ROUGH* and look into further properties of the categories, related to topos and quasitopos. A generalization of the category *RSC* over an arbitrary topos is then introduced and basic properties studied.

Let $\overline{\mathcal{X}}_R$ and $\underline{\mathcal{X}}_R$ denote the collections of equivalence classes of X contained in the R-upper approximation and R-lower approximation of X respectively.

Definition 1. *(ROUGH category [1]) Objects of ROUGH have the form* $\langle U, R, X\rangle$ *(as above). An arrow in ROUGH with domain* $\langle U, R, X\rangle$ *and codomain* $\langle V, S, Y\rangle$ *is a map* $f : \overline{\mathcal{X}}_R \to \overline{\mathcal{Y}}_S$ *such that* $f(\underline{\mathcal{X}}_R) \subseteq \underline{\mathcal{Y}}_S$. *The arrows preserve the lower approximation.*

Definition 2. *(RSC category [2]) Objects of RSC have the form* (X_1, X_2), *where* X_1, X_2 *are sets and* $X_1 \subseteq X_2$. *An arrow in RSC with domain* (X_1, X_2) *and codomain* (Y_1, Y_2) *is a map* $f : X_2 \to Y_2$ *such that* $f(X_1) \subseteq Y_1$.

Proposition 1. *RSC is equivalent to ROUGH.*

Therefore, *RSC* and *ROUGH* share all category-theoretic properties. In [1], it was shown that *ROUGH* is finitely complete and not a topos. [2] showed that *RSC* is not a topos but a weak topos. Further, we have

Proposition 2. *RSC has the following properties.*

1. *The category is finitely cocomplete.*
2. *For the class of monics, relations are not represented.*
3. *An arrow* $m : (X_1, X_2) \to (Y_1, Y_2)$ *is an equalizer, if and only if* m *is monic and* $X_1 = X_2 \cap Y_1$, *and that is if and only if* m *is a strong monic.*
4. *For the class of strong monics, subobject classifier exists.*
5. *For the class of strong monics, partial morphisms are represented.*

Now a category $\mathscr{C}$ is a quasitopos if and only if it is complete, cocomplete, cartesian closed and its partial morphisms are represented for the class of strong monics. Thus we have from Proposition 2,

Theorem 1. *ROUGH, and equivalently RSC, is a quasitopos.* □

Arrows in *ROUGH* preserve the lower approximation. However, some arrows may preserve the boundary, i.e. $\overline{\mathcal{X}}_R \setminus \underline{\mathcal{X}}_R$, as well. The $\xi - ROUGH$ category was defined to capture this property in [1].

Definition 3. *($\xi - ROUGH$ category) Objects are $\langle U, R, X\rangle$, that is same as $ROUGH$. An arrow in $\xi{-}ROUGH$ with domain $\langle U, R, X\rangle$ and codomain $\langle V, S, Y\rangle$ is a map $f : \overline{\mathcal{X}}_R \to \overline{\mathcal{Y}}_S$ such that $f(\underline{\mathcal{X}}_R) \subseteq \underline{\mathcal{Y}}_S$ and $f(\overline{\mathcal{X}}_R \setminus \underline{\mathcal{X}}_R) \subseteq \overline{\mathcal{Y}}_S \setminus \underline{\mathcal{Y}}_S$.*

$\xi - ROUGH$ (a topos) is a subcategory of $ROUGH$ (a quasitopos), both having the same collection of objects. One can define $\xi - RSC$ in the similar way, and observe that

Theorem 2. *$\xi - RSC$, equivalent to $\xi - ROUGH$, is equivalent to the category SET^2, thus forming a topos.*

A generalization of the categories

Let $\mathscr{C}$ be an arbitrary topos, and consider the following category.

Definition 4. *($RSC(\mathscr{C})$ category) Objects of $RSC(\mathscr{C})$ are represented as (A, B) where A and B are objects in $\mathscr{C}$ such that there exist a monic arrow $m : A \to B$ in $\mathscr{C}$. An arrow in $RSC(\mathscr{C})$ with domain (X_1, X_2) and codomain (Y_1, Y_2) is a pair of arrows (f',f) where $f' : X_1 \to Y_1$ and $f : X_2 \to Y_2$ in $\mathscr{C}$ such that $m'f' = fm$, where m and m' are monic arrows corresponding to the objects (X_1, X_2) and (Y_1, Y_2) in $RSC(\mathscr{C})$.*

We observe that $RSC(\mathscr{C})$ is a generalization of the category RSC. A similar construction gives the generalization $\xi - RSC(\mathscr{C})$. In a similar way as we have checked for RSC, various topos and quasitopos-related properties can be established for $RSC(\mathscr{C})$. We obtain

Theorem 3. *$RSC(\mathscr{C})$ is a quasitopos, and $\xi - RSC(\mathscr{C})$ forms a topos.*

Future work

Following are some possible directions of further work. As $ROUGH$ and $\xi - ROUGH$ are quasitopos and topos respectively, they have an inherent logic [3]. It should be worth checking what the involved constructs look like, explicitly, and also put the logics in perspective.

Just as a topos is a category-theoretic abstraction of SET, the category of sets, the quasitopos $RSC(\mathscr{C})$ is an abstraction of the category $ROUGH$ of rough sets. Thus the generalizations proposed here and investigations into their relationships, appear to be significant for the foundations of rough sets from the perspective of category theory.

References

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3. Wyler, O.: *Lecture Notes on Topoi and Quasitopoi*, World Scientific (1991).