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Variable-Precision Compatibility Spaces

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Abstract

In the paper we propose an extension of Wong, Wang, and Yao's rough set model over two universes (called the WWY-rough set model), along the lines of Ziarko's variable-precision rough set model. Within the obtained structures, called the variable-precision compatibility spaces (VPC-spaces), one can reason about compatibility of sets of objects to a varying degree of precision.

Key words: Pawlak's rough set model, variable-precision rough set model, WWY-rough set model, variable-precision compatibility space.

1 Introduction

Wong, Wang, and Yao [19,20,22,23] generalized the Pawlak rough set model [8,9] by considering two universes, U_1 and U_2, related to each other by means of a binary relation $\delta \subseteq U_1 \times U_2$. Let $u \in U_1$ and $v \in U_2$. As mentioned in [22], δ may be understood as a compatibility relation. Under this interpretation, $(u, v) \in \delta$ reads as "u is compatible with v". Along the lines of Pawlak, subsets of the 2nd universe are approximated by means of subsets of the 1st universe, taking compatibility into account. To distinguish the Pawlak lower and upper approximations of sets from their corresponding forms in the the Wong, Wang, and Yao (in short, WWY) rough set model, we shall refer to the latter ones as the WWY-lower and upper approximations.

Within the WWY-rough set model one can reason about various forms of compatibility of objects or sets of objects of possibly two different sorts. A well-known motivating example [22] is a medical diagnosis system, where

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symptoms of diseases are taken as elements of U_1 and diseases – as elements of U_2. Compatibility of a symptom $u \in U_1$ with a disease $v \in U_2$ may be defined as the possibility of occurrence of u in the case of v. Another example is an information system in the sense of Pawlak [8,9], where U_1 is a set of objects and U_2 is a set of attributes. Compatibility of an object u with an attribute v may be understood as relevance of v for u or as accessibility of the value of v for u.

Another motivation underlies the variable-precision (or probabilistic) rough set model proposed by Ziarko [21,27,28]. Instead of representing sets by their lower and upper rough approximations, we can determine the positive, negative, and boundary regions of sets of objects to a varying degree of precision.

In the present paper we extend and refine the WWY-rough set model in the vein of Ziarko's model. To this end we augment their framework with rough inclusion functions [1,11,18]. We next define positive, negative, and boundary regions of sets of objects to a varying degree of precision. Apart from graded approximation of sets of objects of one sort by means of sets of objects of possibly another sort, we can reason about various forms of graded compatibility of sets of objects, e.g., about graded relevance of sets of attributes for sets of objects.

The paper is organized as follows. Section 2 contains preliminaries. In Sect. 3 we concisely describe a slightly generalized version of the WWY-rough set model, where the assumption of finiteness of the universes is dropped. We next recall the notion of a rough inclusion function (Sect. 4). An augmentation of the WWY-rough set model with rough inclusion functions leads to structures called variable-precision compatibility spaces (in short, VPC-spaces), described in Sect. 5. For such structures we define and discuss properties of the counterparts of the positive, negative, and boundary regions of sets of objects. A brief summary is given in Sect. 6.

2 Preliminaries

Throughout the paper we shall use the terms "granule of information" and "granularity mapping". Along the standard lines, an (elementary) granule of information is a set of objects of some sort drawn together and/or towards some object on the base of similarity and/or functionality (cf. Zadeh's definition [24] with modifications proposed by Lin [4]). The computing with granules instead of single pieces of information seems to be a promising way to extract knowledge from an enormous amount of incomplete and imprecise information. Let $u \in U_1$ and $v \in U_2$. In our case two kinds of elementary granules of information will be considered: (a) u is a member of an elementary granule associated with v because u is compatible with v or (b) v is a member of an elementary granule associated with u because u is compatible with v. By a granularity mapping we understand a mapping which assigns granules of information to objects.

The cardinality of a set x will be denoted by $\#(x)$ and the power set of x by $\wp(x)$. Consider any sets x, y, z, w. A relation $\rho \subseteq x \times y$ is *serial* in case $\forall u \in x.\exists v \in y.(u, v) \in \rho$. If $z \subseteq x$ and $w \subseteq y$, then we denote the image of z by $\rho^{\rightarrow}(z)$ and the inverse image of w by $\rho^{\leftarrow}(w)$, respectively. Given mappings $f : x \mapsto y$ and $g : y' \mapsto z$ where $f^{\rightarrow}(x) \subseteq y'$, the composition of f and g is a mapping $g \circ f : x \mapsto z$ such that for every $u \in x$, $(g \circ f)(u) = g(f(u))$. Parentheses will be omitted whenever convenient.

In the paper we shall mainly consider mappings of the forms $f : \wp x \mapsto \wp x$ and $g : \wp x \times y \mapsto \wp x$. Let $\leq$ be an ordering relation on y. f is called *monotone* (resp., *co-monotone*) if for every sets w, z such that $w \subseteq z \subseteq x$, it holds that $f(w) \subseteq f(z)$ (resp., $f(z) \subseteq f(w)$). g is *monotone (co-monotone) in the 1st variable* if for every sets w, z such that $w \subseteq z \subseteq x$ and each $t \in y$, it holds that $f(w,t) \subseteq f(z,t)$ $(f(z,t) \subseteq f(w,t))$. Finally, g is *monotone (co-monotone) in the 2nd variable* if for each $z \subseteq x$ and every $t, s \in y$, $t \leq s$ implies $f(z,t) \subseteq f(z,s)$ $(f(z,s) \subseteq f(z,t))$.

3 The WWY-Rough Set Model

In this section we describe a slightly generalized version of Wong, Wang, and Yao's rough set model (the *WWY-rough set model*) [19,20,22,23]. The WWY-rough set model differs from the Pawlak rough set model [8,9] in that (a) two non-empty finite sets of objects (universes) are considered and (b) subsets of the 2nd universe are approximated by subsets of the 1st one. Let us consider non-empty sets U_i $(i = 1, 2)$ and a mapping $\Delta : U_1 \mapsto \wp U_2$ such that

(1) $$\forall u \in U_1.\Delta(u) \neq \emptyset \text{ and } \bigcup \Delta^{\rightarrow} U_1 = U_2.$$

The latter condition may also be expressed as $\Delta^{\rightarrow} U_1$ is a covering of U_2. In Wong, Wang, and Yao's approach, it is assumed that the universes U_i are finite as well.

Observe that we can define a mapping $\Delta^* : U_2 \mapsto \wp U_1$, associated with Δ, where for any $u \in U_2$,

(2) $$\Delta^*(u) \stackrel{\text{def}}{=} \{v \in U_1 \mid u \in \Delta(v)\}.$$

It is easy to check that Δ^* has analogous properties as Δ. Starting with a serial relation $\delta \subseteq U_1 \times U_2$ such that its converse relation δ^{-1} is serial as well, we can obtain Δ and Δ^* in the following way: For any $u_i \in U_i$ $(i = 1, 2)$, let

(3) $$\Delta(u_1) = \delta^{\rightarrow}\{u_1\} \text{ and } \Delta^*(u_2) = \delta^{\leftarrow}\{u_2\}.$$

Δ is viewed as a granulation mapping which assigns to each $u \in U_1$, a granule $\Delta(u)$ consisting of objects of U_2 that u is in some sense compatible with. Similarly, Δ^* is a granulation mapping which assigns to each $u \in U_2$, a granule $\Delta^*(u)$ consisting of objects of U_1 that are compatible with u. In this way we obtain two families of *elementary granules* of information, $\Delta^{\rightarrow} U_1$ and $\Delta^{*\rightarrow} U_2$, where for any $u \in U_1$ and $v \in U_2$, $\Delta(u)$ and $\Delta^*(v)$ are elementary granules of information drawn towards u and v, respectively, on the base of compatibility of some kind.

Like in the case of generalized approximation spaces [7,11,18], we say that a set $x \subseteq U_2$ is Δ-*definable* iff there is $y \subseteq U_1$ such that $x = \bigcup \Delta^{\rightarrow} y$. Similarly, $x \subseteq U_1$ is Δ^*-*definable* iff there is $y \subseteq U_2$ such that $x = \bigcup \Delta^{*\rightarrow} y$.

To illustrate the above notions, we can think of U_1 as a set of objects and of U_2 as a set of attributes. In particular, if U_1, U_2 are finite, (U_1, U_2) is an information system in the sense of Pawlak [8,9]. With every attribute $a \in U_2$ we can associate a set $V_a \cup \{\bot\}$, where V_a is a non-empty set of proper values of a, while $\bot$ is a constant introduced to speak of (ir)relevance of attributes for objects. Let $V = \bigcup_{a \in U_2} V_a \cup \{\bot\}$. Along the standard lines, we can view attributes as mappings which assign values to objects. [3] Thus, an attribute $a \in U_2$ is a mapping $a : U_1 \mapsto V$ which assigns to every object $u \in U_1$, an element of V_a if a is relevant for u; otherwise, $a(u) = \bot$. $a(u) = \bot$ reads as "a is *irrelevant* for u", while $a(u) \neq \bot$ is understood as "a is *relevant* for u". For instance, the attribute *salary* is usually relevant for objects of the sort *human*. Exceptions are objects being *humans* and *children* for which *salary* is typically irrelevant. However, a slightly different attribute *income* is already relevant for *human children*. *Salary* is typically irrelevant for objects of the sort *animal*. However, in the case of objects of the sort *horse*, *salary* may or may not be relevant. Information systems of the above form resemble incomplete information systems, where missing values are inaccessible (see, e.g., [14] for a survey and [2,12,13,15,17,16] for some works related to our case). It should be emphasized that (ir)relevance and (in)accessibility overlap but in general neither relevance of an attribute causes accessibility of the values of this attribute nor accessibility implies relevance. For instance, the values of the attribute *religion* may be relevant but inaccessible. On the other hand, the values of the attribute *weight* may be accessible but irrelevant.

In the next step we assume that (a) for each $u \in U_1$, there is $a \in U_2$ such that $a(u) \neq \bot$ and (b) for each $a \in U_2$, there is $u \in U_1$ such that $a(u) \neq \bot$. Δ may be defined, for instance, as follows:

$$\Delta(u) \stackrel{\mathrm{def}}{=} \{a \in U_2 \mid a(u) \neq \bot\} \tag{4}$$

where $u \in U_1$. Hence for every $a \in U_2$,

$$\Delta^*(a) \stackrel{\mathrm{def}}{=} \{u \in U_1 \mid a(u) \neq \bot\}. \tag{5}$$

Example 3.1 Let $U_1 = \{u_1, \ldots, u_6\}$ and $U_2 = \{a_1, \ldots, a_5\}$. Relevance of attributes of U_2 for objects of U_1 is described in Tabl. 1. The mappings Δ and Δ^* are given in Tabl. 2 and Tabl. 3, respectively. Next, consider sets of attributes $x_1 = \{a_1, a_3, a_4\}$, $x_2 = \{a_1, a_3, a_5\}$, $x_3 = \{a_2, a_3\}$, $x_4 = \{a_4\}$ and sets of objects $y_1 = U_1 - \{u_4\}$, $y_2 = \{u_1, u_2, u_5\}$, $y_3 = \{u_3, u_6\}$. Sets x_1 and x_2 are Δ-definable since $x_1 = \Delta(u_3)$ and $x_2 = \Delta(u_2) \cup \Delta(u_5)$. Similarly, y_1 is

[3] From another point of view, an object $u \in U_1$ is a mapping $u : U_2 \mapsto V$ which assigns to every attribute $a \in U_2$, an element of V_a if a is relevant for u; otherwise, $u(a) = \bot$. In this case U_1 is a subset of the product $\prod_{a \in U_2}(V_a \cup \{\bot\})$.

Table 1
An exemplary table of relevance of attributes for objects.

$u \backslash a$	a_1	a_2	a_3	a_4	a_5
u_1	$\perp$		$\perp$		
u_2	$\perp$	$\perp$		$\perp$	
u_3		$\perp$			$\perp$
u_4	$\perp$		$\perp$		$\perp$
u_5		$\perp$	$\perp$	$\perp$	$\perp$
u_6	$\perp$				

Table 2
Values of Δ.

u	u_1	u_2	u_3	u_4	u_5	u_6
$\Delta(u)$	$\{a_2, a_4, a_5\}$	$\{a_3, a_5\}$	$\{a_1, a_3, a_4\}$	$\{a_2, a_4\}$	$\{a_1\}$	$U_2 - \{a_1\}$

Table 3
Values of Δ^*.

a	a_1	a_2	a_3	a_4	a_5
$\Delta^*(a)$	$\{u_3, u_5\}$	$\{u_1, u_4, u_6\}$	$\{u_2, u_3, u_6\}$	$\{u_1, u_3, u_4, u_6\}$	$\{u_1, u_2, u_6\}$

Δ^*-definable since $y_1 = \Delta^*(a_1) \cup \Delta^*(a_5)$. The remaining sets are not definable in our sense.

Consider a triple $\mathcal{M} = (U_1, U_2, \Delta)$ as earlier. Along the lines of Wong, Wang, and Yao [19,20,22,23], every set $x \subseteq U_2$ may be approximated by means of subsets of U_1 as follows: Define mappings $\mathrm{low}, \mathrm{upp} : \wp U_2 \mapsto \wp U_1$ such that for any $x \subseteq U_2$,

$$\mathrm{low}(x) \stackrel{\mathrm{def}}{=} \{u \in U_1 \mid \Delta(u) \subseteq x\},$$

(6) $$\mathrm{upp}(x) \stackrel{\mathrm{def}}{=} \{u \in U_1 \mid \Delta(u) \cap x \neq \emptyset\}.$$

$\mathrm{low}(x)$ and $\mathrm{upp}(x)$ play the role of the lower and upper approximations of x in $\mathcal{M}$, respectively. Therefore they will be referred to as the WWY-lower and upper approximations of x, respectively. The pair $(\mathrm{low}, \mathrm{upp})$ is a particular case of an *interval structure*.

low and upp enjoy many of the properties characterizing the Pawlak lower and upper rough approximation mappings, respectively.

Proposition 3.2 *For any set y, a mapping $f : y \mapsto \wp U_1$, any $x \subseteq U_2$, and a family $\{x_j\}_{j \in J}$ of subsets of U_2, we have:*

(a) $\mathrm{low}(\emptyset) = \mathrm{upp}(\emptyset) = \emptyset$ and $\mathrm{low}(U_2) = \mathrm{upp}(U_2) = U_1$.
(b) low and upp are monotone.
(c) $\mathrm{low}(x) \subseteq \mathrm{upp}(x)$.
(d) $\mathrm{low}(x) = U_1 - \mathrm{upp}(U_2 - x)$ and $\mathrm{upp}(x) = U_1 - \mathrm{low}(U_2 - x)$.
(e) $\mathrm{upp}(x) = \bigcup \Delta^{*\rightarrow} x$.
(f) $\forall x \in y. f(x)$ is Δ^* – definable iff $\exists g : y \mapsto \wp U_2. f = \mathrm{upp} \circ g$.
(g) $\mathrm{low}(\bigcup_{j\in J} x_j) \supseteq \bigcup_{j\in J} \mathrm{low}(x_j)$.
(h) $\mathrm{low}(\bigcap_{j\in J} x_j) = \bigcap_{j\in J} \mathrm{low}(x_j)$.
(i) $\mathrm{upp}(\bigcup_{j\in J} x_j) = \bigcup_{j\in J} \mathrm{upp}(x_j)$.
(j) $\mathrm{upp}(\bigcap_{j\in J} x_j) \subseteq \bigcap_{j\in J} \mathrm{upp}(x_j)$.

Proof. We prove (e) and (f) only. For (e) consider $u \in U_1$. $u \in \bigcup \Delta^{*\rightarrow} x$ iff there is $v \in x$ such that $u \in \Delta^*(v)$ iff there is v such that $v \in x$ and $v \in \Delta(u)$ iff $\Delta(u) \cap x \neq \emptyset$ iff $u \in \mathrm{upp}(x)$. For (f) assume that $f(x)$ is Δ^*-definable for each $x \in y$. Hence for every $x \in y$, $\mathcal{Z} = \{z \subseteq U_2 \mid f(x) = \mathrm{upp}(z)\} \neq \emptyset$ in virtue of (e). By the axiom of choice, there is a mapping $g : y \mapsto \wp U_2$ such that for any $x \in y$, $g(x) \in \mathcal{Z}$. Hence $f(x) = (\mathrm{upp} \circ g)(x)$ as needed. The converse implication is obvious. □

Notice that neither $\mathrm{low}(x) \subseteq x$ nor $x \subseteq \mathrm{upp}(x)$ in general. To remove this possible drawback we can take $\mathrm{low}', \mathrm{upp}' : \wp U_2 \mapsto \wp U_2$, defined below, instead of low, upp, respectively. For any $x \subseteq U_2$, let

$$\mathrm{low}'(x) \stackrel{\mathrm{def}}{=} \bigcup \{\Delta(u) \mid u \in U_1 \wedge \Delta(u) \subseteq x\},$$

(7) $$\mathrm{upp}'(x) \stackrel{\mathrm{def}}{=} \bigcup \{\Delta(u) \mid u \in U_1 \wedge \Delta(u) \cap x \neq \emptyset\}.$$

In other words,

$$\mathrm{low}'(x) = \bigcup \Delta^{\rightarrow} \mathrm{low}(x),$$

(8) $$\mathrm{upp}'(x) = \bigcup \Delta^{\rightarrow} \mathrm{upp}(x).$$

One can easily prove the following property:

Proposition 3.3 *For any* $x \subseteq U_2$, $\mathrm{low}'(x) \subseteq x \subseteq \mathrm{upp}'(x)$.

Proof. Consider $u \in U_2$. If $u \in \mathrm{low}'(x)$, then there is $v \in U_1$ such that $u \in \Delta(v)$ and $\Delta(v) \subseteq x$. Next suppose that $u \in x$. By definition, there is $v \in U_1$ such that $u \in \Delta(v)$. Clearly $\Delta(v) \cap x \neq \emptyset$. Hence $u \in \mathrm{upp}'(x)$ as needed. □

Example 3.4 (A continuation of Example 3.1.) Table 4 contains the WWY-lower and upper approximations of x_i $(i = 1, \ldots, 4)$ and their modified versions.

Table 4
The WWY-lower and upper approximations of exemplary sets.

x	x_1	x_2	x_3	x_4
$\mathrm{low}(x)$	$\{u_3, u_5\}$	$\{u_2, u_5\}$	$\emptyset$	$\emptyset$
$\mathrm{upp}(x)$	U_1	$U_1 - \{u_4\}$	$U_1 - \{u_5\}$	$\{u_1, u_3, u_4, u_6\}$
$\mathrm{low}'(x)$	$\{a_1, a_3, a_4\}$	$\{a_1, a_3, a_5\}$	$\emptyset$	$\emptyset$
$\mathrm{upp}'(x)$	U_2	U_2	U_2	U_2

4 Rough Inclusion Functions

Informally speaking, rough inclusion functions are functions which assign to every pair of sets of objects (x, y), a number of the unit interval $[0, 1]$ expressing the degree of inclusion of x in y. Several various rough inclusion functions are proposed in the literature [1,11,18,27]. The best known is the *standard* rough inclusion function (cf. [6,27]) $\kappa^s : \wp U \times \wp U \mapsto [0, 1]$, where U is a non-empty finite set and for any $x, y \subseteq U$,

(9) $$\kappa^s(x, y) \stackrel{\text{def}}{=} \begin{cases} \frac{\#(x \cap y)}{\#x} & \text{if } x \neq \emptyset \\ 1 & \text{otherwise.} \end{cases}$$

If U is infinite, the above function may be undefined if the first argument is infinite. However, the assumption of finiteness of the 2nd argument of κ^s may be dropped.

In our approach, a *rough inclusion function* is a mapping $\kappa : \wp U \times \wp U \mapsto [0, 1]$ such that for any sets $x, y, z \subseteq U$, the following conditions are satisfied:

(κ1) $\kappa(x, y) = 1$ iff $x \subseteq y$.
(κ2) If $x \neq \emptyset$, then $\kappa(x, y) + \kappa(x, U - y) = 1$.
(10) (κ3) If $y \subseteq z$, then $\kappa(x, y) \leq \kappa(x, z)$.

(κ1) says that the degree of inclusion of x in y in the rough sense is the greatest iff x is included in y in the classical sense. Hence $\kappa(\emptyset, y) = 1$ for every $y \subseteq U$. In connection with (κ2) we can ask for which $t \in [0, 1]$ it is possible that $\kappa(x, y) \geq t$ and $\kappa(x, U - y) \geq t$, provided that $x \neq \emptyset$. Suppose that $\kappa(x, y) \geq t$ and $\kappa(x, U - y) \geq t$. If $t > 0.5$, then $\kappa(x, y) + \kappa(x, U - y) > 1$ contrary to (κ2). Nevertheless, we can answer the above question positively if $t \leq 0.5$. The first two conditions imply that if x is non-empty, then the degree of rough inclusion of x in y is the least iff the intersection of x and y is empty, i.e.,

(11) $$\text{if } x \neq \emptyset, \text{ then } \kappa(x, y) = 0 \text{ iff } x \cap y = \emptyset.$$

Hence for non-empty sets x, y, $\kappa(x,y) = 0$ iff $\kappa(y,x) = 0$. The condition ($\kappa 2$) is reasonable but strong. As suggested by Andrzej Skowron and Dominik Ślęzak, it would be interesting to consider a weaker version of it. ($\kappa 3$) expresses monotonicity of κ in the 2nd variable. κ may be neither monotone nor co-monotone in the 1st variable. Indeed, it can be $x \subseteq y$ and $\kappa(x,z) \not\leq \kappa(y,z)$. To see this take x, y, z such that $x \subseteq y \cap z$ and $y \not\subseteq z$. Then $\kappa(x,z) = 1$ and $\kappa(y,z) < 1$. Similarly, it can hold $x \subseteq y$ and $\kappa(y,z) \not\leq \kappa(x,z)$. Namely, let $\emptyset \neq x \subseteq y$, $x \cap z = \emptyset$, and $y \cap z \neq \emptyset$.

5 A Variable-Precision Compatibility Model

An extension of the WWY-rough set model with rough inclusion functions, as proposed in this section, results in a variable-precision rough set model over two universes along the lines of Ziarko [21,27,28]. Let $i = 1, 2$, U_i be non-empty sets, $\Delta : U_1 \mapsto \wp U_2$ be a granulation mapping, and $\kappa_i : \wp U_i \times \wp U_i \mapsto [0,1]$ be rough inclusion functions, described earlier. By a *variable-precision compatibility space (VPC-space)* we mean a tuple of the form $\mathcal{M} = (U_1, U_2, \Delta, \kappa_1, \kappa_2)$.

We now define the variable-precision positive, negative, and boundary regions of sets of objects along the lines of Ziarko [21,27,28]. Let $i = 1, 2$, $t, s_i \in [0,1]$, $s = (s_1, s_2)$, $s_1 < s_2$, and $x \subseteq U_2$. By $\mathrm{pos}(x,t)$, $\mathrm{neg}(x,t)$, and $\mathrm{bnr}(x,s)$ we denote the *t-positive region*, the *t-negative region*, and the *s-boundary region* of x, respectively, defined as follows:

$$\mathrm{pos}(x,t) \stackrel{\mathrm{def}}{=} \bigcup\{\Delta(u) \mid u \in U_1 \wedge \kappa_2(\Delta(u), x) \geq t\}.$$

$$\mathrm{neg}(x,t) \stackrel{\mathrm{def}}{=} \bigcup\{\Delta(u) \mid u \in U_1 \wedge \kappa_2(\Delta(u), x) \leq t\}.$$

$$\text{(12)}\quad \mathrm{bnr}(x,s) \stackrel{\mathrm{def}}{=} \bigcup\{\Delta(u) \mid u \in U_1 \wedge s_1 < \kappa_2(\Delta(u), x) < s_2\}.$$

Let us observe that

$$\text{(13)}\qquad \mathrm{low}'(x) = \mathrm{pos}(x,1).$$

Consider a non-empty family $\{t_j\}_{j \in J}$ of numbers of the unit interval. By $\inf_{j \in J}\{t_j\}$ (resp., $\sup_{j \in J}\{t_j\}$) we understand the minimal (resp., maximal) element of the family if it exists, or 0 (resp., 1) otherwise. Basic properties of pos, neg, and bnr are presented below.

Proposition 5.1 *For x, t, s_1, s_2, s as earlier, any family $\{t_j\}_{j \in J}$ of numbers of $[0,1]$, and any family $\{x_j\}_{j \in J}$ of subsets of U_2, we have:*

(a) $\mathrm{neg}(x,t) = \mathrm{pos}(U_2 - x, 1 - t)$.
(b) $\mathrm{bnr}(x,s) = U_2 - \mathrm{neg}(x, s_1) - \mathrm{pos}(x, s_2)$.
(c) $\mathrm{pos}(x,0) = \mathrm{neg}(x,1) = U_2$.
(d) If $t > 0$, then $\mathrm{pos}(\emptyset, t) = \emptyset$.
(e) If $t < 1$, then $\mathrm{neg}(U_2, t) = \emptyset$.
(f) $\mathrm{pos}(U_2, t) = \mathrm{neg}(\emptyset, t) = U_2$.

(g) pos is monotone in the 1st variable and co-monotone in the 2nd variable.

(h) neg is co-monotone in the 1st variable and monotone in the 2nd variable.

(i) $\mathrm{pos}(x,1) \subseteq x$ and $\mathrm{neg}(x,0) \subseteq U_2 - x$.

(j) $\mathrm{pos}(\bigcup_{j\in J} x_j, t') \supseteq \bigcup_{j\in J} \mathrm{pos}(x_j, t_j)$ where $t' = \inf_{j\in J}\{t_j\}$.

(k) $\mathrm{pos}(\bigcap_{j\in J} x_j, t') \subseteq \bigcap_{j\in J} \mathrm{pos}(x_j, t_j)$ where $t' = \sup_{j\in J}\{t_j\}$.

Proof. We prove (j) only. If $J = \emptyset$, the property holds trivially. Assume that $J \neq \emptyset$. Consider $u \in U_2$. $u \in \bigcup_{j\in J} \mathrm{pos}(x_j, t_j)$ iff there are $j \in J$ and $v \in U_1$ such that $u \in \Delta(v)$ and $\kappa_2(\Delta(v), x_j) \geq t_j$. By (10), $\kappa_2(\Delta(v), x_j) \leq \kappa_2(\Delta(v), \bigcup_{j\in J} x_j)$. On the other hand, $t' \leq t_j$. Hence $u \in \mathrm{pos}(\bigcup_{j\in J} x_j, t')$. □

Now let $x \subseteq U_1$. In the same vein, we can define the *$t*$-positive region* of x, written $\mathrm{pos}^*(x,t)$, the *$t*$-negative region* of x, $\mathrm{neg}^*(x,t)$, and the *$s*$-boundary region* of x, $\mathrm{bnr}^*(x,s)$, viz.,

$$\mathrm{pos}^*(x,t) \stackrel{\mathrm{def}}{=} \bigcup\{\Delta^*(u) \mid u \in U_2 \wedge \kappa_1(\Delta^*(u), x) \geq t\},$$

$$\mathrm{neg}^*(x,t) \stackrel{\mathrm{def}}{=} \bigcup\{\Delta^*(u) \mid u \in U_2 \wedge \kappa_1(\Delta^*(u), x) \leq t\},$$

$$\mathrm{bnr}^*(x,s) \stackrel{\mathrm{def}}{=} \bigcup\{\Delta^*(u) \mid u \in U_2 \wedge s_1 < \kappa_1(\Delta^*(u), x) < s_2\}. \quad (14)$$

To obtain properties for pos^*, neg^*, and bnr^* it suffices to replace U_1 by U_2, and vice versa in Proposition 5.1.

Example 5.2 (A continuation of Example 3.1.) Let $\kappa_1 = \kappa_2 = \kappa^s$. In the VPC-space $\mathcal{M} = (U_1, U_2, \Delta, \kappa_1, \kappa_2)$, for some degrees of precision $t \in [0,1]$ and $s \in [0,1]^2$, we compute the t-positive regions of $x_1, \ldots, x_4$ and the $t*$-positive regions of y_1, y_2, y_3 (Tabl. 5), the t-negative regions of $x_1, \ldots, x_4$ and the $t*$-negative regions of y_1, y_2, y_3 (Tabl. 6), and finally, the s-boundary regions of $x_1, \ldots, x_4$ and the $s*$-boundary regions of y_1, y_2, y_3 (Tabl. 7).

6 Final Remarks

In the paper we generalized and refined the Wong, Wang, and Yao's rough set model over two universes by introducing positive, negative, and boundary regions of sets of objects to a varying degree of precision. Our research is still in progress yet. For instance, we hope to further explore the illustrating example of an information system, where (ir)relevance of attributes for objects is taken into account.

Table 5
Examples of the t- and $t*$-positive regions of sets.

$z\backslash t$	0.3	0.4	0.5	0.6	0.7	1.0
x_1	U_2	U_2	U_2	$\{a_1, a_3, a_4\}$	$\{a_1, a_3, a_4\}$	$\{a_1, a_3, a_4\}$
x_2	U_2	U_2	U_2	$U_2 - \{a_2\}$	$\{a_1, a_3, a_5\}$	$\{a_1, a_3, a_5\}$
x_3	U_2	$U_2 - \{a_1\}$	$U_2 - \{a_1\}$	$\emptyset$	$\emptyset$	$\emptyset$
x_4	U_2	$\{a_2, a_4\}$	$\{a_2, a_4\}$	$\emptyset$	$\emptyset$	$\emptyset$
y_1	U_1	U_1	U_1	U_1	U_1	$U_1 - \{u_4\}$
y_2	U_1	$U_1 - \{u_4\}$	$U_1 - \{u_4\}$	$\{u_1, u_2, u_6\}$	$\emptyset$	$\emptyset$
y_3	U_1	U_1	U_1	$\{u_2, u_3, u_6\}$	$\emptyset$	$\emptyset$

Table 6
Examples of the t- and $t*$-negative regions of sets.

$z\backslash t$	0.2	0.3	0.4	0.6	0.7
x_1	$\emptyset$	$\emptyset$	$\{a_2, a_4, a_5\}$	$U_2 - \{a_1\}$	$U_2 - \{a_1\}$
x_2	$\{a_2, a_4\}$	$\{a_2, a_4\}$	$\{a_2, a_4, a_5\}$	$U_2 - \{a_1\}$	U_2
x_3	$\{a_1\}$	$\{a_1\}$	U_2	U_2	U_2
x_4	$\{a_1, a_3, a_5\}$	U_2	U_2	U_2	U_2
y_1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{u_1, u_4, u_6\}$
y_2	$\emptyset$	$\{u_1, u_3, u_4, u_6\}$	$U_1 - \{u_5\}$	U_1	U_1
y_3	$\emptyset$	$\emptyset$	$\{u_1, u_2, u_4, u_6\}$	U_1	U_1

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Table 7
Examples of the s- and $s*$-boundary regions of sets.

$z \backslash s$	$(0.2, 0.7)$	$(0.3, 0.6)$	$(0.4, 0.7)$	$(0.4, 1.0)$	$(0.6, 1.0)$
x_1	$\{a_2, a_5\}$	$\{a_2, a_5\}$	$\emptyset$	$\emptyset$	$\emptyset$
x_2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_3	$U_2 - \{a_1\}$	$U_2 - \{a_1\}$	$\emptyset$	$\emptyset$	$\emptyset$
x_4	$\{a_2, a_4\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
y_1	$\emptyset$	$\emptyset$	$\emptyset$	$\{u_4\}$	$\{u_4\}$
y_2	U_1	$\{u_5\}$	$\{u_5\}$	$\{u_5\}$	$\emptyset$
y_3	U_1	$\{u_1, u_4, u_5\}$	$\{u_3, u_5\}$	$\{u_3, u_5\}$	$\emptyset$

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