Rough Sets and Information Granulation

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Abstract. In this paper, the study of the evolution of approximation space theory and its applications is considered in the context of rough sets introduced by Zdzisław Pawlak and information granulation as well as computing with words formulated by Lotfi Zadeh. Central to this evolution is the rough-mereological approach to approximation of information granules. This approach is built on the inclusion relation to be a part to a degree, which generalises the rough set and fuzzy set approaches. An illustration of information granulation of relational structures is given. The contribution of this paper is a comprehensive view of the notion of information granule approximation, approximation spaces in the context of rough sets and the role of such spaces in the calculi of information granules.

Keywords: Approximation spaces, calculus of granules, information granulation, rough mereology, rough sets.

1 Introduction: Rough Set Approach to Concept Approximation

One of the basic concepts of rough set theory [7] is the indiscernibility relation defined by means of information about objects. The indiscernibility relation is used to define set approximations [6, 7]. There have been reported several generalisations of the rough set approach based, e.g., on approximation spaces defined by tolerance and similarity relation, or a family of indiscernibility relations (for references see the papers and bibliography in [5, 12]). Rough set approximations have also been generalised for preference relations and rough-fuzzy hybridisations (see, e.g., [21]). Let us consider a generalised approximation spaces introduced in [18]. It is defined by $AS = (U, \mathcal{N}, \nu)$ where U is a set of *objects* (universe), $\mathcal{N}$ is an *uncertainty function* defined on U with values in the powerset $P(U)$ of U (e.g. $\mathcal{N}(x)$ can be interpreted as a *neighbourhood* of x), and ν is an *inclusion function* defined on the Cartesian product $P(U) \times P(U)$ with values in the interval $[0, 1]$

(or more generally, in a partially ordered set), measuring the degree of inclusion of sets. In the sequel $\nu_p(X,Y)$ denotes $\nu(X,Y) \geq p$ for $p \in [0,1]$. The lower AS_* and upper AS^* approximation operations can be defined in AS by

$$AS_*(X) = \{x \in U : \nu(\mathcal{N}(x), X) = 1\}, \quad (1)$$

$$AS^*(X) = \{x \in U : \nu(\mathcal{N}(x), X) > 0\}. \quad (2)$$

The neighbourhood of an object x can be defined by the indiscernibility relation IND. If IND is an equivalence relation then we have $\mathcal{N}(x) = [x]_{IND}$. In the case where IND is a tolerance (similarity) relation $\tau \subseteq U \times U$, we take $\mathcal{N}(x) = \{y \in U : x\tau y\}$, i.e., $N(x)$ is equal to the tolerance class of τ defined by x. The standard inclusion function is defined by $\nu(X,Y) = \frac{|X \cap Y|}{|X|}$ if X is non-empty and by $\nu(X,Y) = 1$ otherwise. In addition, parameterisation of $\mathcal{N}$ and ν in an AS makes it possible to calibrate (tune) approximation operations, and leads to a form of learning in rough neural networks designed in the context of approximation spaces (see, e.g., [16, 10]). For applications it is important to have some constructive definitions of $\mathcal{N}$ and ν. The approach based on inclusion functions has been generalised to the *rough mereological approach* (see, e.g., [11, 13, 14]). The inclusion relation $x\mu_r y$ with the intended meaning *x is a part of y to a degree r* has been taken as the basic notion of the rough mereology which is a generalisation of the Leśniewski mereology [1].

We consider two sources of information granulation. The first source is related to inductive reasoning and the second source arises due to object indiscernibility.

As a result of inductive reasoning one cannot define inclusion degrees of object neighbourhoods directly into the target concepts but only into some patterns relevant to such concepts (e.g. left hand sides of decision rules) (see [16, 23]). Such degrees together with degrees of inclusion of patterns in target concepts make it possible to define outputs of information granules, called classifiers, for new objects.

In case of indiscernibility of objects it may be necessary, for example, to consider more general structures of the uncertainty function. The values of an uncertainty function may belong to a more complex set than $P(U)$ (see, e.g., [2, 4]). We will illustrate this case using an example of a relational structure granulation.

2 Rough-mereological Approach to Approximation of Information Granules

Rough mereology offers a methodology for synthesis and analysis of objects in the distributed environments of intelligent agents, in particular, for synthesis of objects satisfying a given specification to a satisfactory degree, or for control in such complex environments. Moreover, rough mereology has been recently used for developing foundations of the *information granule calculus*, an attempt towards formalisation of the paradigm of computing with words based on perception, recently formulated by Lotfi Zadeh [24–26].

The rough mereological approach is built on the basis of the inclusion relation *to be a part to a degree* and generalises the rough set and fuzzy set approaches (see. e.g., [11, 13–15]). Such a relation is called *rough inclusion*. This relation can be used to define other basic concepts like closeness of information granules, their semantics, indiscernibility and discernibility of objects, information granule approximation and approximation spaces, perception structure of information granules as well as the notion of ontology approximation. For details the reader is referred to [4]. The rough inclusion relations together with operations for construction of new information granules from already existing ones create a core of a calculus of information granules.[1] A distributive multi-agent framework makes it possible to create a relevant computational model for a calculus of information granules. Agents (information sources) provide us with information granules that must be transformed, analysed and built into structures that support problem solving. In such a computational model, approximation spaces play an important role because information granules received by agents must be approximated (to be understandable by them) before they can be transformed (see, e.g., [13, 19, 4, 10]).

Developing calculi of information granules for approximate reasoning is a challenge important for many applications including control of autonomous vehicles [22] and line-crawling robots [8], web mining and spatio-temporal data mining [16], design automation, sensor fusion [9], approximation neuron design [10, 4], creation of approximate views of relational databases, and, in general, for embedding in intelligent systems the ability to reason with words as well as perception based reasoning [24–26]. Some steps towards this direction have been taken. Methods for construction of approximate reasoning schemes (*AR*-schemes) have been developed. Such *AR*-schemes are information granules that are clusters of exact constructions (derivations). Reasoning with *AR*-schemes makes it possible to obtain results satisfying a given specification up to a satisfactory degree (i.e. not necessarily exactly) (see e.g. [13, 4, 16, 17]). Methods based on hybridisation of rough sets with fuzzy sets, neural networks, evolutionary approach or case based reasoning are especially valid in inducing *AR*-schemes.

Let us note that inducing relevant calculi of information granules includes also such complex tasks like discovery of relevant operations on information granules or rough inclusion measures. This is closely related to problems of perception and reasoning based on perception [26].

Using rough inclusions, one can generalise the approximation operations for sets of objects, known in rough set theory, to arbitrary information granules. The approach is based on the following reasoning:

Assume $G = \{g_t\}_t$ is a given family of information granules, g is a given granule, and p, q are inclusion degrees such that $p < q$ (let us recall that inclusion degrees are partially ordered by a relation $\leq$). One can consider two

[1] Note, the rough inclusion relations should be extended on newly constructed information granules.

kinds of approximation of granule g by a family of granules G. The (G, q)-*lower approximation* of g is defined by[2]

$$LOW_{G,q}(g) = Make\_granule(\{g_t : \nu(g_t, g) \geq q\}). \tag{3}$$

The (G, p)-*upper approximation* of g is defined by

$$UPP_{G,p}(g) = Make\_granule(\{g_t : \nu(g_t, g) > p\}). \tag{4}$$

The definition of a generalised approximation space defined in section 1 is a special case of the notion of information granule approximation. The presented approach can be generalised to approximation spaces in inductive reasoning.

3 Illustrative Example: Granulation of Relational Structures

In this section we present an illustrative example of information granulation. Among the basic concepts, to be used here, is a relational structure M of a given signature Sig with a domain Dom and a language L of signature Sig. The neighbourhood (uncertainty) function is then any function $\mathcal{N} : Dom \rightarrow P^{\omega}(Dom)$ where

- $P^{\omega}(Dom) = \bigcup_{k \in \omega} P^k(Dom)$.
- $P^0(Dom) = Dom$ and $P^{k+1}(Dom) = P(P^k(Dom))$ for any non-negative integer k.

To explain this concept let us consider an information system $\mathbb{A} = (U, A)$ as an example of relational structure. A neighbourhood function $\mathcal{N}_{\mathbb{A}}$ of $\mathbb{A}$ is defined by $\mathcal{N}_{\mathbb{A}}(x) = [x]_{IND(A)}$ for $x \in U = Dom$ where $[x]_{IND(A)}$ denotes the A-indiscernibility class of x. Hence, the neighbourhood function forms basic granules of knowledge about the universe U. Let us consider a case where the values of neighbourhood function are from $P^2(Dom)$. Assume that together with an information system $\mathbb{A}$ there is also given a similarity relation τ defined on vectors of attribute values. This relation can be extended to objects. An object $y \in U$ is similar to a given object $x \in U$ if the attribute value vector on x is τ-similar to the attribute value vector on y. Now, consider a neighbourhood function defined by $\mathcal{N}_{\mathbb{A},\tau}(x) = \{[y]_{IND(A)} : x\tau y\}$.

Neighbourhood functions cause a necessity of further granulation. Let us consider granulation of a relational structure M by neighbourhood functions. We would like to show that due to the relational structure granulation we obtain new information granules of more complex structure and in the consequence more general neighbourhood functions than those discussed above. Hence, basic

[2] *Make_granule* operation is a fusion operation of collections of information granules. A typical example of *Make_granule* is set theoretical union used in rough set theory. Another example of *Make_granule* operation is realised by classifiers.

granules of knowledge about the universe corresponding to objects become more complex.

Assume that a relational structure M and a neighbourhood function $\mathcal{N}$ are given. The aim is to define a new relational structure $M_{\mathcal{N}}$ called the $\mathcal{N}$–granulation of M.[3] This is done by granulation of all components of M by means of $\mathcal{N}$. Let us assume $M_{\mathcal{N}} \supseteq P(Dom)$. We present examples showing that such domain $M_{\mathcal{N}}$ should consist of elements of $P^{\omega}(Dom)$.

Let us consider a binary relation $r \subseteq Dom \times Dom$. There are numerous possible ways to define a $\mathcal{N}$–granulation $r_{\mathcal{N}}$ of relation r – the choice depends on applications. Let us list some possible examples:

$$\begin{aligned} & r_{\mathcal{N}}(\mathcal{N}(x), \mathcal{N}(y)) \text{ iff } \mathcal{N}(x) \times \mathcal{N}(y) \subseteq r \\ & r_{\mathcal{N}}(\mathcal{N}(x), \mathcal{N}(y)) \text{ iff } (\mathcal{N}(x) \times \mathcal{N}(y)) \cap r \neq \emptyset \\ & r_{\mathcal{N}}(\mathcal{N}(x), \mathcal{N}(y)) \text{ iff } card\,((\mathcal{N}(x) \times \mathcal{N}(y)) \cap r) \geq s \cdot card\,(\mathcal{N}(x)) \\ & \text{where } s \in [0,1] \text{ is a threshold.} \end{aligned} \tag{5}$$

In this way some patterns are created for pairs of objects. Such patterns can be used for approximation of a target concept (or concept on an intermediate level) over objects composed from pairs (x, y). Certainly, to induce approximations of high quality it is necessary to search for relevant patterns for concept approximation expressible in a given language. This problem is discussed, e.g., in [20].

Let us consider an exemplary degree structure $D = ([0,1], \leq)$ and its granulation $D_{\mathcal{N}_0} = (P([0,1]), \leq_{\mathcal{N}_0})$ by means of an uncertainty function $\mathcal{N}_0 : [0,1] \rightarrow P([0,1])$ defined by $\mathcal{N}_0(x) = \{y \in [0,1] : [10^k x] = [10^k y]\}$, for some integer k, where for $X, Y \subseteq [0,1]$ we assume $X \leq_{\mathcal{N}_0} Y$ iff $\forall x \in X, \forall y \in Y\ x \leq y$. Let $\{X_s, X_m, X_l\}$ be a partition of $[0,1]$ satisfying $x < y < z$ for any $x \in X_s$, $y \in X_m$, $z \in X_l$. Let $AS_0 = ([0,1], \mathcal{N}_0, \nu)$ be an approximation space with the standard inclusion function ν. We denote by S, M, L the lower approximations of X_s, X_m, X_l in AS_0, respectively, and by S–M, M–L the boundary regions between X_s, X_m and X_m, X_l, respectively. Moreover, we assume $S, M, L \neq \emptyset$. In this way we obtain restriction of $D_{\mathcal{N}_0}$ to the structure $(Deg, \leq_{\mathcal{N}_0})$, where $Deg = \{S, S\text{–}M, M, M\text{–}L, L\}$. Now, for a given (multi-sorted) structure $(U, P(U), [0,1], \leq, \mathcal{N}_0, \nu)$, where $\nu : P(U) \times P(U) \rightarrow [0,1]$ is an inclusion function, we can define its $\mathcal{N}_0$–granulation by

$$(U, P(U), Deg, \leq_{\mathcal{N}_0}, \{\nu_d\}_{d \in Deg}) \tag{6}$$

where $Deg = \{S, S\text{–}M, M, M\text{–}L, L\}$ and $\nu_d(X,Y)$ iff $\nu_p(X,Y)$ for some p, d', such that $p \in d'$ and $d \leq_{\mathcal{N}_0} d'$.

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[3] In general, granulation is defined using the uncertainty function and the inclusion function from a given approximation space. For simplicity, we restrict our initial examples to $\mathcal{N}$–granulation only.

Now, let us consider a function f from M and some possible $\mathcal{N}$-granulations $f_{\mathcal{N}}$ of f.[4]

$$f_{\mathcal{N}}(\mathcal{N}(x), \mathcal{N}(y)) = \{\mathcal{N}(z) : z = f(x', y') \text{ for some } x' \in \mathcal{N}(x), y' \in \mathcal{N}(y)\} \tag{7}$$

$$f_{\mathcal{N}}(\mathcal{N}(x), \mathcal{N}(y)) = \bigcup_{x' \in \mathcal{N}(x), y' \in \mathcal{N}(y)} \{\mathcal{N}(z) : z = f(x', y')\}$$

$$f_{\mathcal{N}}(\mathcal{N}(x), \mathcal{N}(y)) = \bigcup_{x' \in \mathcal{N}(x), y' \in \mathcal{N}(y)} \{\mathcal{N}(z) : card\,(\mathcal{N}(z)) \geq s \text{ and } z = f(x', y')\}$$

where $s \in [0, 1]$ is a threshold.

The values of $f_{\mathcal{N}}$ can be treated as generators for patterns used for the target concept approximation. An example of a pattern language can be obtained by considering the results of set theoretical operations on neighbourhoods. Observe that the values of the function $f_{\mathcal{N}}$ are in $P^2(Dom)$. Hence, one could extend the neighbourhood function and the relation granulation on this more complex domain. Certainly this process can be continued and more complex patterns can be generated. On the other hand it is also necessary to bound the depth of exploration of $P^{\omega}(Dom)$. This can be done by using the rough set approach. For example, after generation of patterns from $P^2(Dom)$ one should, in a sense, reduce them to $P(Dom)$ by considering some operations from $P^2(Dom)$ into $P(Dom)$ returning the relevant patterns for the target concept approximation. Such reduction is necessary especially if the target concepts are elements of the family $P(Dom)$.

In general, relational structures are granulated by means of a given approximation space. Thus, we use in granulation both the uncertainty and the inclusion functions (see, e.g., (6)). Observe, that approximation spaces are also (multi-sorted) relational structures. Hence, they can be also granulated in searching for relevant approximation spaces for concept approximations. Let us consider an example of granulation of a given approximation space $AS = (U, P(U), [0, 1], \leq, \mathcal{N}, \nu)$ (in most of the cases we use simplified notation $(U, \mathcal{N}, \nu)$). The granulation of AS is defined by using AS itself. The resulting approximation space is then $AS' = (P(U), \mathcal{N}_p, \nu')$ where

1. for some $p \in [0, 1]$ the uncertainty function $\mathcal{N}_p : P(U) \to P^2(U)$ is defined by $\mathcal{N}_p(X) = \{Y \in P(U) : \nu_p(X, Y) \text{ and } \nu_p(Y, X)\}$, i.e., $\mathcal{N}_p(X)$ is a cluster of sets close to X to degree at least p;
2. for $\mathcal{X}, \mathcal{Y} \in P^2(U)$ and $q \in [0, 1]$ we assume $\nu'_q(\mathcal{X}, \mathcal{Y})$ iff for any $X \in \mathcal{X}$ there exists $Y \in \mathcal{Y}$ such that $\nu_q(X, Y)$.

One can consider another granulation of the inclusion functions assuming for $\mathcal{X}, \mathcal{Y} \in P^2(U)$ and $q \in [0, 1]$

$$\nu'_q(\mathcal{X}, \mathcal{Y}) \text{ iff } \frac{card((AS_p)_*(\mathcal{X}) \cap (AS_p)_*(\mathcal{Y}))}{card((AS_p)_*(\mathcal{X}))} > q$$

[4] For simplicity, we assume f has two arguments and values of $\mathcal{N}$ are in $P(Dom)$.

where $AS_p = (P(U), \mathcal{N}_p, \nu)$ is an approximation space with the standard inclusion function $\nu : P^2(U) \times P^2(U) \to [0, 1]$.

In multi-agent setting [16, 17, 19, 20] each agent is equipped with its own relational structure and approximation spaces located in input ports. The approximation spaces are used for filtering (approximating) information granules sent by other agents. Such agents are performing operations on approximated information granules and sending the results to other agents, checking relationships between approximated information granules, or using such granules in negotiations with other agents. Parameterised approximation spaces are analogous to weights in classical neurons. Agents are performing operations on information granules (approximating concepts) rather than on numbers. This analogy has been used as a starting point for the rough-neuro computing paradigm [4].

4 Conclusions

We have discussed approximation spaces developed and investigated in the context of rough set theory and the role of such spaces in the calculi of information granules. Generalised approximation spaces have been briefly considered. These spaces include an uncertainty function as well an inclusion function defined relative to a set of objects called a universe. The basic idea of a calculus of granules rooted in a rough mereological approach has been presented in the context of information granulation approximation. It has also been suggested how one might create patterns that can be used to approximate target concepts. Pattern generators have also been considered. This research provides promising avenues for the study of pattern generation and the discovery of patterns for target concept approximation.

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