Chapter 3

Rough Mereology in Information Systems. A Case Study: Qualitative Spatial Reasoning

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Abstract. Rough Mereology has been proposed as a paradigm for approximate reasoning in complex information systems [64], [65], [66], [67], [75]. Its primitive notion is that of a rough inclusion functor which gives for any two entities of discourse the degree in which one of them is a part of the other. Rough Mereology may be regarded as an extension of Rough Set Theory as it proposes to argue in terms of similarity relations induced from a rough inclusion instead of reasoning in terms of indiscernibility relations; it also proposes an extension of Mereology as it replaces the mereological primitive functor of being a part with a more general functor of being a part in a degree. Rough Mereology has deep relations to Fuzzy Set Theory as it proposes to study the properties of partial containment which is also the fundamental subject of study for Fuzzy Set Theory. Rough Mereology may be regarded as an independent first order theory (such point of view was adopted in [64], [65], [66], [67], [75]) but it may be also formalized in the traditional mereological scheme proposed by Stanisław Leśniewski where Mereology is regarded and formalized within Ontology i.e. Theory of Names (Concepts). In this Chapter we take the latter course. We regard this approach as particularly suited for Rough Set Theory which is also primarily concerned with Concept Approximation in Information Systems.

In this Chapter, we give a description of Rough Mereology in Information Systems along the lines outlined above : we give an introduction to Ontology and Mereology according to Leśniewski and we show how one may introduce them in Information Systems on the basis of Rough Set Theory. In this framework, we introduce Rough Mereology and we present some ways for defining rough inclusions. We demonstrate applications of Rough Mereology to approximate reasoning taking as the case subject Qualitative Spatial Reasoning. This topic seems to be especially suitable for rough mereological approach as it relies very essentially on Ontology and Mereology; we address its mereotopological as well

as geometrical aspects.

keywords: *rough sets, spatial reasoning, rough mereology, ontology, mereology, information systems/tables*

1 Introduction

Rough Mereology has been proposed as a tool for reasoning under uncertainty (approximate reasoning) with data collected in information systems as well as a general paradigm within which it would be possible to formally describe schemes for synthesis of approximate solutions to problems posed uncertainly, vaguely or incompletely [64], [65], [66], [67], [75]. It has been shown to constitute a general framework (cf. Chapter 3: A Perspective, in ([68], vol.1) in which it is possible to develop a theory of rough computation with applications to distributed computing, knowledge granulation and computing with words (cf. [98], [99]).

Two guiding paradigms for Rough Mereology were: the Theory of Rough Sets [61] and the Theory of Fuzzy Sets [100]. From Rough Set Theory inherits Rough Mereology the idea of approximation of general concepts with particular i.e. exact concepts: in Rough Set Theory exact concepts are defined by means of attribute–value descriptors and a fortiori they are finite unions of indiscernibility classes with the indiscernibility relations providing partitions of the universe of objects. In recent investigations the need for more relaxed approximations, preferably induced by similarity or tolerance relations has been stressed and experimentally verified to yield better classification results [78], [74], [87], [73], [8], [40].

Rough Mereology proposes as its primitive notion the notion of a *rough inclusion* which is a parameterized functor μ_r such that for any pair of individual entities X, Y the formula $Y \varepsilon \mu_r(X)$ means that Y is a part of X in degree r where $r \in [0,1]$; the rough inclusion may be regarded thus as a parameterized family of similarity relations: fixing r, we may define a similarity relation sim_r viz. $X sim_r Y$ if and only if $Y \varepsilon \mu_r(X)$. One may then define approximations of concepts using the family sim_r along the lines of Rough Set Theory with modifications described e.g. in [87].

As any rough inclusion is concerned with relations among objects expressed by means of degrees of partness of objects, Rough Mereology has clear connections to Fuzzy Set Theory whose basic subject of study are partial containment as well as partial membership cf. [100].

Yet another source of ideas and points of reference for Rough Mereology are Mereological Theories of Sets. We refer here to two mainstream theories of Mereology viz. Mereology due to Stanisław Leśniewski [51], [53], [54], [83], [84], [20], [89], [15], [50] and Mereology based on Connection [14], [48], [95], [17], [18], [57], [4], [6].

Of the two theories, Mereology based on Connection offers a richer variety of mereotopological functors; yet, as Mereology of Leśniewski is based on the notion of *part*, it offers a formalism of which the formalism of Rough Mereology may be –under a suitable choice of primitive expressions–a direct extension and generalization. This is actually the case: Rough Mereology was proposed [64], [65], [66], [67] to contain Mereology of Leśniewski as the theory of the functor μ_1 and this feature is preserved in the formalization proposed here.

In the scheme envisioned by Stanisław Leśniewski, Mereology was to follow Ontology i.e. the Theory of Names (Concepts) and with this purpose in mind he proposed his Ontology [37], [49], [52], [79]. Ontological theories play an important role in Approximate Reasoning [12], [34], [80] witnessed with particular clearness in Spatial Reasoning [56], [24] where Ontology plays a basic role as it sets spatial concepts and their taxonomy; for the same reason Ontology is an immanent, although frequently implicit, component of Rough Set Theory as the latter discusses concepts and their approximations hence it is vitally interested in taxonomies of concepts and relations among them.

From this place a twofold way stretches forth; first, we may try to localize Rough Mereology within Ontology in particular within Ontology of Information Systems (i.e. Rough Set Ontology) and second, we may use Rough Mereology to create Ontology for a particular domain of interest. Here we select as such Case Study domain the domain of Spatial Reasoning because of the eminent role played in its development by mereology–based methods.

In this Chapter we take both ways throughout the text and they meet again in the final sections where we discuss Rough Mereotopology and Rough Mereogeometry in the context of Spatial Reasoning in Information Systems.

Let us observe here in passing that rough set–theoretic ideas have already been applied in problems of Spatial Reasoning e.g. in problems of multi–resolution spatial reasoning [86], [97], in problems of localization (rough localization) [10], and in the *egg–yolk approach* [19], where a region with uncertain boundary is enclosed in two regions with definite boundaries, one may find ideas very close to rough set–theoretic ideas of approximation. This fact prompts us even more towards a discussion of synthesis of concepts pertaining to Spatial Reasoning by means of Rough Mereology.

We propose to begin in Section 2 with a brief introduction to Rough Set Theory with emphasis on the nature of concepts encountered therein. Then we propose to introduce in respective Sections 3 and 4 Ontology, respectively, Mereology of Leśniewski along with interpretations of these theories in Information Systems.

Rough Mereology is discussed in Section 5 with examples of rough inclusions induced in information systems for various ontologies discussed in Section 3.

Section 6 introduces basic aspects of Qualitative Spatial Reasoning and in Section 7 devoted to Mereotopology we show that rough inclusions induce quasi–Čech topologies which in certain models become naturally quasi–topologies (i.e. topologies without the null element).

Section 8 introduces connection functors induced from rough inclusions; we demonstrate the invariance of the element functor *el* under those connections and we relate topologies induced by rough inclusions to topologies generated from connections induced by those rough inclusions.

The final Section 9 outlines how one may define basic primitives of geometry by means of rough inclusions and this Chapter concludes in Section 10 with examples concerning rough merotopology as well as rough mereogeometry.

2 Rough Set Theory

Rough Set Theory has to do with data represented as an *information system*; the information system is formally presented as a pair A=(U, A) where U is a universe of objects (represented as rows in the *information table* $U \times A$) and A is a set of *attributes* (represented as columns in the *information table* $U \times A$).

Each attribute $a \in A$ is formalized as a function $a : U \longrightarrow V_a$ where V_a is the *value set* of a. We may assume that for any a, the values in the a–column in the information table exhaust V_a.

The ontological assumption about the information system (U, A) is that it does represent the whole content of knowledge about the real world to which the data refer. A fortiori, any inference about the world is to be made on the basis of the knowledge represented by the given information system and what bears on the knowledge content is actually not individual objects (rows) but rather their information sets.

For an object $u \in U$, we call an *information set* of u, the set $Inf_A(u) = \{(a, a(u)) : a \in A\}$ (when no confusion arises, we write simply $Inf(u)$ or v_A when mentioning u is not necessary).

In consequence of the above assumption, we identify any two objects (rows) whose information sets are identical; formally, we define the *A–indiscernibility relation* IND_A as follows: $(u, u') \in IND_A \Longleftrightarrow Inf_A(u) = Inf_A(u')$.

Then, the A-indiscernibility classes $[u]_a$ are represented by information sets $Inf_A(u)$ in the one–one way, and the information system (U, A) may be reduced to the information system (V, A) where V=$\{Inf_a(u) : u \in U\}$ and $a(v)$= $a(u)$ where $v = Inf_A(u)$ for any $v \in V$ and any $a \in A$.

A basic Ontology of an information system may be based on individuals being objects in U i.e. rows of the data table and general names being *concepts* i.e. sets of individual entities. Among concepts one has to make a distinction: there are concepts which may be described in terms of attributes and their values completely and certainly and there are concepts whose description is by necessity uncertain and incomplete.

To make this distinction more clear, we first observe that the notions of an information set as well as of the indiscernibility relation may be taken relative to a set of attributes: given a non–empty set $B \subset A$ of attributes, we define for any $U \in U$ its *B–information set* $Inf_B(u)$ as the set $\{(a, a(u)) : a \in B\}$ and the *B–indiscernibility relation* IND_B is defined accordingly: $(u, u') \in IND_B \Longleftrightarrow Inf_B(u) = Inf_B(u')$.

Now, given $B \subseteq A$ $(B \neq \emptyset)$, we may define a *B–exact concept* as a concept X such that X may be represented as a union of B–indiscernibility classes i.e. **if** $u \in X \wedge (u, u') \in IND_B$ **then** $U' \in X$.

An A–*elementary exact concept* will be defined now as a B-exact concept for some B. Returning to B, we may now describe all concepts approximately in terms of B–attributes and their values, viz. given a non–empty concept $X \subseteq U$, we define:

the B–lower approximation $B_{-}X = \{u \in U : [u]_B \subseteq X\}$;
the B–upper approximation $B^{+}X = \{u \in U : [u]_B \cap X \neq \emptyset\}$.

For each concept X, one may now describe X approximately–by means of knowledge represented by the set B of attributes– as the pair $(B_{-}X, B^{+}X)$ of two B–exact sets. The general properties of this approximation may be found in [61].

Clearly, a concept X is a B–exact concept if and only if the condition $B_{-}X = B^{+}X$ holds. Otherwise, X is said to be a *B–inexact concept.* More generally, we may extend this definition taking into account all classes $[u]_B$, any B : for a concept X, we let

the A*–lower approximation* $\mathsf{A}_{-}X = \{u \in U : \exists B.[u]_B \subseteq X\}$;
the A*–upper approximation* $\mathsf{A}^{+}X = \{u \in U : \forall B.[u]_B \cap X \neq \emptyset\}$.

We will call a concept X an A*–exact concept* in case $\mathsf{A}_{-}X = \mathsf{A}^{+}X$; otherwise, X will be called an A*–inexact concept.* Clearly, any elementary A–exact concept is an A–exact concept.

The taxonomy of concepts outlined above may be also rendered by means of algebraic, topological or logical structures. To this end, let us observe that

Proposition 1. *The set–theoretic operations of the union, the intersection and the complement preserve* A*–exact sets as well as B–exact sets for any B.*

Let us recall, that given a universe U, a family F of subsets of U closed under the set–theoretic operations of the union, the intersection and the complement is called a ***field of sets***; any field of sets is a special case of a ***boolean algebra*** i.e. a set endowed with the operations of the join $\vee$, the meet $\wedge$ and the complement $'$ having formal properties analogous to those of $\cup$, $\cap$, $-$ cf. [71]. An ***atom*** in a boolean algebra $(U, \vee, \wedge, ')$ is any non–zero u with the property that $\forall v \neq 0. u \wedge v = v \Longleftrightarrow u = v$ i.e. u is a minimal non–zero element with respect to inclusion cf. [71].

A boolean algebra is *atomic* if and only if any non–zero element contains an atom. Thus, Proposition 1 may be restated in an algebraic form cf. [13], [60]:

Proposition 2. *1. For any B, the set of all B–exact sets with the operations of the union, the intersection and the complement is an atomic boolean algebra with atoms being B–indiscernibility classes;*

2. The set of all A*–exact sets with operations of the union, the intersection and the complement is an atomic boolean algebra with atoms being B–indiscernibility classes for any B.*

Similarly, topological paraphrase follows. Recall that a topological space cf. [46] is a set U along with a family O of its subsets closed under any union, finite intersection and containing the empty set. Sets in O are *open sets*; *closed* sets then are complements to open sets. A topology may be induced into a set U also by means of operators called the *interior*, resp. the *closure* op.cit.: the interior of a set X is the union of all elements of O contained in X while the closure of X is the intersection of all closed sets containing X. A set X which is open as well as closed is called a *clopen*. Now, we may restate Propositions 1 and 2 and earlier discussion as follows cf. [77], [96].

Proposition 3. *1. For any B, the set of all B–exact sets is a topological space in which any B–exact set is a clopen;*
2. The set of all A–exact sets is a topological space in which any A–exact set is a clopen;
3. For any B, the lower approximation operator B_- is the interior operator in the topological space of all subsets of U; similarly, the upper approximation operator B^+ is the closure operator in that topological space. The same is true for approximations A_-, A_+, respectively.

Finally, one may give a logical frame to our discussion cf. [61], [76]. Let us call a *descriptor* any pair of the form (a, v) where $a \in A$ and $v \in V_a$; descriptors may be regarded as elementary formulae from which formulae of the *descriptor logic* are formed by means of the propositional connectives $\vee$ and $\wedge$ (let us note that the negation is not necessary in the finite case considered here). The meaning $[\alpha]$ of a formula α is defined inductively:

$[(a, v)] = \{u \in U : a(u) = v\}$;
$[\alpha \vee \beta] = [\alpha] \cup [\beta]$;
$[\alpha \wedge \beta] = [\alpha] \cap [\beta]$.

For any pair $B \subseteq A, v = Inf_B(u)$, where $u \in U$, the formula $\alpha_{B,v}$ is the conjunction $\wedge_{a \in B}(a, v_a)$; clearly, $[\alpha_{B,v}] = [u]_B$.

In the sequel, we will call a pair (B, v) as above a *template*. Templates (defined also equivalently as $\alpha_{B,v}$) play an important role in rough set–theoretic methods in Knowledge Discovery and Data Mining cf. a throughout discussion in [74], [73], [8].

In the above discussion, we actually have introduced several types of concepts (names) like an A–exact set, B–exact set, etc. and we have also some examples of objects (represented as subsets of U) which answer to some of these names. For instance, when $U = \{u_1, u_2\}$, $A = \{a_1, a_2\}$ and $V_a = \{0, 1\}$ for each a, then the set $X = \{u : a_1(u) = 0\}$ is both A–exact and B–exact where $B = \{a_1\}$. In an informal way, we would write these facts down using phrases of the form: "X *is* B–exact" etc.

Similar situation happens in other contexts where approximate reasoning is carried out within other paradigms; e.g. in developing schemes for Qualitative Spatial Reasoning (see below), one comes at a very early stage at the necessity to introduce concepts, or, names, for spatial entities in question, and to set relations of hierarchy among those names i.e. at the necessity for Ontology. In application-oriented spatial reasoning systems, ontology appears as typology of concepts and their successive taxonomy cf. e.g. [56] (to quote a small excerpt: *edge* is *frontier*, *barrier*, *dam*, *cliff*, *shoreline*).

It would be therefore profitable for our discussion, if we could introduce a formal Ontology fit for our purpose of presenting rough mereology and its applications. So we now give a formal scheme of Ontology and next we show that it does agree with our taxonomy presented above. The ontological scheme we choose to apply here is the Leśniewski Ontology cf. [37], [49], [52], [79].

We first give an introduction to formal Ontology and then we propose some interpretations of this formal scheme in Information Systems.

The reader may throughout the formal part of the next Section think of a formula $X\varepsilon Y$ as true if and only if X is a name of an individual element and Y is a name of a set of elements (actually, our specific ontologies are of this type).

3 Ontology: An Introduction to the Leśniewski Ontology

Ontology was intended by Stanisław Leśniewski as a formulation of general principles of being [52] cf. also [38], [49], [79], [37].

The only primitive notion of Ontology of Leśniewski is the copula "*is*" denoted by the symbol ε.

All well–formed expressions of Ontology belong in classes called semantic categories. Categories are constructed starting with the two basic semantic categories: the semantic category of names and the semantic category of propositions. Either of those categories does contain constants as well as variables of the given category.

Higher order categories are constructed as functors under the agreement that each functor does belong either in the category of names (a *name– forming* functor) or in the category of propositions (a *proposition –forming* functor); however, a stratification of functors in a usual way is achieved by assigning functors to the same category if both are aither name–forming or proposition–forming and if they have the same number of arguments falling into same categories, respectively.

We denote with capitals $X, Y, Z...$ nominal variables of Ontology; similarly, $\alpha, \beta, ...$ will denote propositional variables.

We will use standard symbols for propositional constants as well as quantifier, parentheses etc. symbols.

3.1 Axiom of Ontology

The original axiom of Ontology defining the meaning of ε is as follows

The ontological axiom

$$X\varepsilon Y \iff \exists Z.Z\varepsilon X \wedge \forall U,W.(U\varepsilon X \wedge W\varepsilon X \Longrightarrow U\varepsilon W) \wedge \forall Z.(Z\varepsilon X \Longrightarrow Z\varepsilon Y).$$

In this axiom, the defined copula ε happens to occur in both sides of the equivalence: however, the definiendum $X\varepsilon Y$ belongs in the left side only and we may perceive the axiom as the definition of the meaning of $X\varepsilon Y$ via the meaning of terms of "lower level" $Z\varepsilon X$, $Z\varepsilon Y$ etc.

According to this reading of the axiom, we gather that the proposition $X\varepsilon Y$ is true if and only if the conjunction holds of the following three propositions:

1. $\exists Z.Z\varepsilon X$; this proposition asserts the existence of an object (name) Z which is X and so X is not an empty name.
2. $\forall U,W.(U\varepsilon X \wedge W\varepsilon X \Longrightarrow U\varepsilon W)$; this proposition asserts that any two objects which are X are each other (a fortiori, they will be identified later on): thus means that X is an individual name (or, X is an individual entity, representable as a singleton).
3. $\forall Z.(Z\varepsilon X \Longrightarrow Z\varepsilon Y)$; this proposition asserts that every object which is X is Y as well (or, X is contained in Y).

The meaning of $X\varepsilon Y$ can be made clear now: X is an individual and this individual is Y (i.e. belongs in Y).

Remark. Assume that we have a family F of (finite, non–empty) sets. Then we may construct a model for elementary Ontology by expressing the functor ε as the inclusion functor $\subset$. Individual objects are then singletons. The formula $X\varepsilon Y$ may be rendered as $\exists a.X = \{a\} \wedge X \subseteq Y$. Clearly, in this context we may introduce the structure of an atomic boolean algebra of sets (with singletons as atoms) by introducing functors of union, intersection and difference of sets with the provision that these operations are not accessible in cases when they would result in an empty set. We will see below that this is also the general case.

The development of Ontology rests now with consequences to the Ontology Axiom derived according to rules of derivation and rules of category formation. We first recall basic scheme of Elementary Ontology. Our presentation is based in large part on [79].

3.2 Elementary Ontology

We will apply here standard rules for adding and omitting the general as well as the existential quantifier in the calculus of predicates viz.
$\frac{\forall x.\alpha(x)}{\alpha(y)}$; $\frac{\exists x.\alpha(x)}{\alpha(z)}$ where z has not yet occurred in the proof; $\frac{\alpha(x)}{\forall x.\alpha(x)}$; $\frac{\alpha(z)}{\exists x.\alpha(x)}$.

Rules for functor formation Rules for functor formation are as follows:

for proposition–forming functors
$f(X_1, X_2, ..., X_n) \iff \alpha$
where α is a propositional expression of ontology;

for name-forming functors

$X\varepsilon f(X_1, X_2, ..., X_n) \iff X\varepsilon X \wedge \alpha$

where α is a propositional expression of ontology;

for nominal constants

$X\varepsilon C \iff X\varepsilon X \wedge \alpha$

where α is a propositional expression of ontology.

As basic examples, we introduce two nominal constants:

Definition 4. 1. X$\varepsilon\Lambda \iff X\varepsilon X \wedge non(X\varepsilon X)$; the constant Λ is the *empty name.*

2. $X\varepsilon V \iff \exists Y.X\varepsilon Y$; the constant V is the *universal name.* $X\varepsilon V$ reads as "X is an object".

Basic facts of elementary ontology We begin with stating the three basic implications which follow from the Axiom of Ontology.

Proposition 5. *1.* $X\varepsilon Y \Longrightarrow \exists Z.Z\varepsilon X$;

2. $X\varepsilon Y \Longrightarrow \forall U, W.(U\varepsilon X \wedge W\varepsilon X \Longrightarrow U\varepsilon W)$;

3. $X\varepsilon Y \Longrightarrow \forall Z.(Z\varepsilon X \Longrightarrow Z\varepsilon Y)$.

We now introduce a name forming functor $\subseteq$ via

Definition 6. $X\varepsilon \subseteq (Y) \iff X\varepsilon X \wedge \forall Z.(Z\varepsilon X \Longrightarrow Z\varepsilon Y)$

We will read $X\varepsilon \subseteq (Y)$ as "X is *contained* in Y".
The basic properties of the functor $\subseteq$ are summarized below.

Proposition 7. *1.* $X\varepsilon \subseteq (X)$;

2. $X\varepsilon \subseteq (Y) \wedge Y\varepsilon \subseteq (Z) \Longrightarrow X\varepsilon \subseteq (Z)$;

3. $X\varepsilon \subseteq (V)$.

Proof. The property (1) follows from the tautology $p \Longrightarrow p$ via instantiation $Z\varepsilon X \Longrightarrow Z\varepsilon X$ and universal quantification: $\forall Z.Z\varepsilon X \Longrightarrow Z\varepsilon X$. Similarly, (2) follows by omitting the general quantifier in the premises $X\varepsilon \subseteq (Y), Y\varepsilon \subseteq (Z)$,applying the inference rule $\frac{U\varepsilon X \Longrightarrow U\varepsilon Y, U\varepsilon Y \Longrightarrow U\varepsilon Z}{U\varepsilon X \Longrightarrow U\varepsilon Z}$ and quantifying universally. For (3), we observe that $Z\varepsilon X \Longrightarrow \exists Y.Z\varepsilon Y \Longrightarrow Z\varepsilon V$.

We introduce yet another name forming functor via

Definition 8. $X\varepsilon = (Y) \iff X\varepsilon X \wedge Y\varepsilon Y \wedge X\varepsilon Y \wedge Y\varepsilon X$

The functor $=$ is the individual identity functor ("X is *identical* to Z"). With its help, we may write down the individuality condition as : $\forall U, W.(U\varepsilon X \wedge W\varepsilon X \Longrightarrow U = W)$.

We state the basic properties of the existential statement $X\varepsilon Y$.

Proposition 9. *1.* $X\varepsilon Y \wedge Z\varepsilon X \Longrightarrow Z\varepsilon Y$;
2. $X\varepsilon Y \Longrightarrow X\varepsilon X$;
3. $X\varepsilon Y \wedge Z\varepsilon X \Longrightarrow Z = X$;
4. $X\varepsilon X \Longleftrightarrow X\varepsilon V$.

Proof. Indeed, (1) follows as $X\varepsilon \subseteq (Y)$ hence $Z\varepsilon X$ implies $Z\varepsilon Y$. (2) , (3) follow by virtue of definitions. Finally, $X\varepsilon X \Longrightarrow X\varepsilon V$ is obvious and $X\varepsilon V \Longrightarrow X\varepsilon X$ patterns (2).

On a higher level of generality, the counterpart of the identity functor $=$ is the scope equality functor $=_E$ defined as follows.

Definition 10. $X =_E Y \Longleftrightarrow \forall Z.(Z\varepsilon X \Longleftrightarrow Z\varepsilon Y)$

We have the following properties of this notion.

Proposition 11. *1.* $X =_E Y \Longleftrightarrow X\varepsilon \subseteq (Y) \wedge Y\varepsilon \subseteq (X)$;
2. $X = Y \Longleftrightarrow X\varepsilon Y \wedge X =_E Y$;
3. $X\varepsilon V \Longrightarrow (X = Y \Longleftrightarrow X =_E Y)$.

All these properties follow immediately from definitions.

Finally, we give, following [79], a logical (extensional) content to individual identity. Let $\alpha(Z)$ be a propositional expression. We denote by $\alpha(X/Z)$ the expression formed from α by replacing Z with X. Then

Proposition 12. $X = Y \Longrightarrow (\alpha(X/Z) \Longleftrightarrow \alpha(Y/Z))$

Proof. (Slupecki) Define a name-forming functor $f(Y, U, W, ..)$ via $Z\varepsilon f(Y, U, W, ..) \Longleftrightarrow Z\varepsilon Z \wedge \alpha(Z)$. Assume that $X = Y$ and $\alpha(X/Z)$; hence $Y\varepsilon X$ and $X\varepsilon\ f(Y, U, W, ..)$ implying $Y\varepsilon f(Y, U, W, ..)$ and $\alpha(Y/Z)$. The conclusion follows by symmetry.

We may now pass to non-elementary ontology.

3.3 Non–Elementary Ontology

In this extension, we may encounter functors of higher order i.e. functors having as arguments also functors.We state additional rules and axioms.

The rule of substitution (cf. [79]).

In expressions of ontology, one may substitute for variables of any semantic category either variables or constants of the same semantic category.

Non-elementary ontology deals with functors of higher orders whose arguments are functors as well. In original expositions of ontology, the difference among functors of various orders was stressed by different notation and symbolics e.g usage of brackets of various shapes. Here, we do not pay attention to these distinction, stressing rather the basic principles.

In addition to the usual rule of adding definitions to the system at any stage, non-elementary ontology uses the other rule: the Extensionality rule which guarantees that functors applied to the arguments already found identical give identical objects. Formally, this rule may be formulated as follows.

The Extensionality Rule Assume that Φ is a proposition–forming functor of nominal variables $\varphi_1, ..., \varphi_k$ and that $\forall x. x\varepsilon\varphi_i \iff x\varepsilon\psi_i$ for $i = 1, 2, .., k$ and nominal variables $\psi_1, ..., \psi_k$; then, $\Phi(\varphi_1, ..., \varphi_k) \iff \Phi(\psi_1, ..., \psi_k)$; in case $\varphi_1, ..., \varphi_k, \psi_1, ..., \psi_k$ are propositional variables the extensionality rule is: $\forall i. \varphi_i \iff \psi_i \implies \Phi(\varphi_1, ..., \varphi_k) \iff \Phi(\psi_1, ..., \psi_k)$. In case Φ is a name forming functor, the consequent of the rule is of the form $\forall x. x\varepsilon\Phi(\varphi_1, ..., \varphi_k) \iff x\varepsilon\Phi(\psi_1, ..., \psi_k)$.

3.4 Basic theorems of non-elementary ontology

Here also we follow [79] in presenting an outline of non–elementary Ontology.

In non–elementary Ontology it is possible to introduce metastatements.

We begin with a statement that any proposition–forming functor induces a name.

Definition 13. For a proposition–forming functor φ, we define a name– forming functor prop(φ) via $X\varepsilon prop(\varphi) \iff X\varepsilon V \wedge \varphi(X)$

Using the functor $prop(\varphi)$, we may prove the *law of identity for extensionality.*

Proposition 14. $X = Y \wedge \varphi(X) \implies \varphi(Y)$

Proof. $X = Y$ hence $X\varepsilon Y$ so $X\varepsilon V$; as $\varphi(X)$ we have $X\varepsilon prop(\varphi)$ and $Y\varepsilon X$ implies $Y\varepsilon prop(\varphi)$ which implies $\varphi(Y)$.

Obviously

Proposition 15. $X = Y \implies \varphi(X) \iff \varphi(Y)$

Conversely, every name induces a proposition–forming functor.

Definition 16. For a name X, let ε_X be a proposition–forming functor defined as $\varepsilon_X(Y) \iff Y\varepsilon X$.

We may prove the converse to the last proposition.

Proposition 17. $X = Y \wedge \forall \varphi . \varphi(X) \iff \varphi(Y) \implies X = Y$.

Proof. Let $X = Y$; as $\forall \varphi . \varphi(X) \iff \varphi(Y)$ we have $\varepsilon_X(X) \iff \varepsilon_X(Y)$ i.e. $X \varepsilon X \iff Y \varepsilon X$ hence $Y \varepsilon X$. Similarly, $Y \varepsilon X$ i.e. $X = Y$.

As a corollary, the ***Leibnizian principle of indiscernibility*** follows cf.[79].

Corollary 18. $X = Y \iff X \varepsilon V \wedge Y \varepsilon V \wedge \forall \varphi . (\varphi(X) \iff \varphi(Y))$.

To produce the counterparts for non–individual names, we need more name–induced functors.

Definition 19. For a name X, let $\varepsilon^X(Y) \iff X \varepsilon Y$.

Then,

Proposition 20. $\forall \varphi . (\varphi(X) \iff \varphi(Y)) \implies X =_E Y$.

Proof. By premise, $\varepsilon^Z(X) \iff \varepsilon^Z(Y)$ i.e. $Z \varepsilon X \iff Z \varepsilon Y$ hence $\forall Z . Z \varepsilon X \iff Z \varepsilon Y$ i.e. $X =_E Y$.

To prove the converse, we need an instance of the extensional rule.

Proposition 21. $\forall Z . (Z \varepsilon X \iff Z \varepsilon Y) \implies \forall \varphi . (\varphi(X) \iff \varphi(Y))$.

Then a corollary follows

Corollary 22. $X =_E Y \iff \forall \varphi . (\varphi(X) \iff \varphi(Y))$.

Counterparts for name-forming functors may be proved similarly.

Definition 23. For a name-forming functor f, we let $\varepsilon(f, X)(Y) \iff X \varepsilon f(Y)$.

We can now prove

Proposition 24. $X =_E Y \implies \forall f . (f(X) =_E f(Y))$.

Proof. As $X =_E Y$, we have by Corollary 22 $\varepsilon(f, Z)(X) \iff \varepsilon(f, Z)(Y)$ i.e. $Z \varepsilon f(X) \iff Z \varepsilon f(Y)$ so $\forall Z . (Z \varepsilon f(X) \iff Z \varepsilon f(Y))$ i.e. $(f(X) =_E f(Y))$.

The converse follows in a similar way via the functor $\varepsilon(Z)$ defined as $\varepsilon(Z)(X) \iff Z \varepsilon X$.

Proposition 25. $\forall f . (f(X) =_E f(Y)) \implies X =_E Y$.

Indeed, by the premises, we have $\varepsilon(Z)(X) \iff \varepsilon(Z)(Y)$ i.e. $X =_E Y$.

Corollary 26. $X =_E Y \iff \forall f.(f(X) =_E f(Y))$.

By applying a reduction technique based on the formal equivalence $(\varphi, X_1)(X_2, .., X_n) \iff \varphi(X_1, X_2, .., X_n)$ and inducting on the arity n,we may prove a general statement

Proposition 27. $\forall_i.(X_i =_E Y_i) \iff \forall f.(f(X_1, X_2, .., X_n) =_E f(Y_1, Y_2, .., Y_n))$.

The above facts correspond to indiscernibility principles of Rough Set Theory when we regard rows in data table as objects and attributes as functors.

Calculus of relations In ontology one may define the notions of a relation, a function, the equipotence etc.cf. [79] which we briefly recapitulate here.

Definition 28. $\mathrm{rel}(\varphi) \iff (\varphi(X_1, X_2, ..., X_n) \implies X_1 \varepsilon V \wedge X_2 \varepsilon V \wedge .... \wedge X_n \varepsilon V)$

Defines a functor *rel* stating that φ is a relation of arity n.
The notions of the domain and co-domain are introduced easily.

Definition 29. 1. $X \varepsilon dom(\varphi) \iff X \varepsilon V \wedge rel(\varphi) \wedge \exists Y.\varphi(X, Y)$.
2. $Y \varepsilon co - dom(\varphi) \iff Y \varepsilon V \wedge rel(\varphi) \wedge \exists X.\varphi(X, Y)$.

The definition of a function is straightforward

Definition 30. $\mathrm{func}(\varphi) \iff rel(\varphi) \wedge \forall X, Z, U.(\varphi(X, Z) \wedge \varphi(X, U) \implies Z = U)$.

Defines the functor of being a function.
The functor of being an injective function may be defined similarly.

Definition 31. $\mathrm{inj}(\varphi) \iff \mathrm{func}(\varphi) \wedge \forall X, Z, U.(\varphi(X, Z) \wedge \varphi(U, Z) \implies X = U)$.

The functor of equipotency follows.

Definition 32. $\mathrm{equip(X,Y)} \iff \exists\varphi.\mathrm{inj}(\varphi) \wedge X =_E dom(\varphi) \wedge Y =_E co - dom(\varphi)$.

An example of new name–forming functors may be given by means of ***fusion operations*** cf. [89], [79], [37]:
given names X, Y, we define new names: $X + Y$, $X \cdot Y$, $X - Y$ as follows:

1. $Z \varepsilon X + Y \iff Z \varepsilon X \vee Z \varepsilon Y$;
2. $Z \varepsilon X \cdot Y \iff Z \varepsilon X \wedge Z \varepsilon Y$;
3. $Z \varepsilon X - Y \iff Z \varepsilon X \wedge non(Z \varepsilon Y)$.

One may notice the similarity of these operations with the operations of the union, the intersection and the difference yielding a field of sets or more generally with the operations of the join, the meet and the complement, leading to a boolean algebra. It is actually the case with Ontology and we will present a demonstration of this fact but in a slightly different context viz. we include a discussion of some equivalent axiom schemes for Ontology.

3.5 Equivalent Axiom schemes

It turns out that one may axiomatize Ontology by means of other notions a fortiori by means of distinct schemes of axioms. The most important are axiomatic characterizations of Ontology by means of ordering functors (relations) as they lead directly to well-known mathematical structures viz. quasi–boolean algebras cf. also [37], [79].

We define a binary proposition–forming functor $ord(X, Y)$ as follows.

Definition 33. $ord(X,Y) \iff \exists Z.Z\varepsilon X \wedge \forall Z.(Z\varepsilon X \implies Z\varepsilon Y)$.

Then we have

Proposition 34. $X\varepsilon Y \iff ord(X,Y) \wedge \forall U,W.(ord(U,X) \wedge ord(W,X) \implies ord(U,W))$.

For a better visualization of the formulae, we introduce a shortcut notation $X < Y$ for $ord(X, Y)$. This is however only a notational convenience, not a new type of a functor.

Proof. Assume first that $X\varepsilon Y$; then clearly $X < Y$. Now assume that $U < X, W < X$. Observe that $U < X$ implies that U is an individual i.e. $U\varepsilon U$ hence $U\varepsilon X$; similarly, $W\varepsilon X$ and as X is an individual, it follows that $U = W$. From $U\varepsilon V, W\varepsilon V$ it follows by Proposition 11 that $U =_E W$ and finally $U < W$.

Now assume that $ord(X,Y) \wedge \forall U,W.(ord(U,X) \wedge ord(W,X) \implies ord(U,W))$. It suffices to check that $U\varepsilon X \wedge W\varepsilon X \implies U\varepsilon W$. Assume $U\varepsilon X \wedge W\varepsilon X$; then $U < X$, $W < X$ and thus $U = W$ implying $X\varepsilon X$ hence $X\varepsilon Y$.

Proposition 35. *[Lejewski [49]]* $\exists Z.Z\varepsilon X \iff \exists Z.Z < X$

Proof. Clearly, $\exists Z.Z\varepsilon X$ implies $\exists Z.Z < X$. Conversely, $\exists Z.Z < X$ gives $A\varepsilon X$, some A hence $\exists B.B\varepsilon A$ and $\forall T.(T\varepsilon A \implies T\varepsilon X)$ so $B\varepsilon X$ and finally $\exists Z.Z\varepsilon X$.

The following proposition establishes a deeper parallellism between "is" and "entails".

Proposition 36. *[Sobociński cf. [49]]*

$$\forall U,W.(U\varepsilon X \wedge W\varepsilon X \implies U\varepsilon W) \iff \forall U,W.(U < X \wedge W < X \implies U < W)$$

Proof. Assume $\forall U,W.(U\varepsilon X \wedge W\varepsilon X \implies U\varepsilon W)$ and $U < X \wedge W < X$. For any P, it suffices to show that $P\varepsilon U \implies P\varepsilon W$. Let $P\varepsilon U$; $Q\varepsilon W$; then, $P\varepsilon X$, $Q\varepsilon X$ and by the premises, $P\varepsilon Q$ hence $P\varepsilon W$.

Conversely, from $\forall U,W.(U < X \wedge W < X \implies U < W)$ and $U\varepsilon X \wedge W\varepsilon X$ it follows that $U < X, W < X$ and so $U < W$; by Proposition , $U\varepsilon W$.

We may now establish

Proposition 37. *[Lejewski [49]]*

$$X < Y \iff \exists Z.Z < X \wedge \forall Z.(Z < X \implies \exists W.(W\varepsilon Z \wedge W\varepsilon Y))$$

Proof. First, let $X < Y$. Then, $\exists Z.Z < X$ hence $\forall T.(T\varepsilon Z \implies T\varepsilon Y)$ and $\exists W.W\varepsilon Z$ which yields $A\varepsilon Z$ hence $A\varepsilon Y$ for some A and finally $\exists W.W\varepsilon Z \wedge W\varepsilon Y$.

Now, let $\exists Z.Z < X \wedge \forall Z.(Z < X \implies \exists W.(W\varepsilon Z \wedge W\varepsilon Y))$; assume $Z\varepsilon X$ so $Z < X$. Then $W\varepsilon Z \wedge W\varepsilon Y$ for some W hence $Z\varepsilon Y$. It follows that $\forall Z.(Z\varepsilon X \implies Z\varepsilon Y)$ so $X < Y$.

The following is an immediate consequence of the above propositions and Definition 33.

Proposition 38. *[Lejewski [49]]*
(ATB) $X < Y \iff \exists Z.Z < X \wedge \forall Z.(Z < X \implies$
$\exists W.(W < Z \wedge W < Y) \wedge \forall P,Q.(P < W \wedge Q < W \implies P < Q))$

Proposition 39. *[Lejewski [49]] (ATB) is equivalent to the Axiom of Ontology*

Proof. First, we show that (ATB) implies the Ontology Axiom. It is sufficient to check that: $(Z < X \implies \exists W.W\varepsilon Z \wedge W\varepsilon Y) \iff (Z\varepsilon X \implies Z\varepsilon Y)$. Assume then that $(Z < X \implies \exists W.W\varepsilon Z \wedge W\varepsilon Y)$ and $Z\varepsilon X$. Hence $Z < X$ and so $W\varepsilon Z, W\varepsilon Y$ for some W and thus $Z\varepsilon Y$. Conversely, from $(Z\varepsilon X \implies Z\varepsilon Y)$ and $Z < X$ it follows $\exists W.W\varepsilon Z$ and $\forall W.(W\varepsilon Z \implies W\varepsilon X)$ hence for some W we have $W\varepsilon Z$ hence $W\varepsilon X$ and finally $W\varepsilon Y$ proving $\exists W.W\varepsilon Z \wedge W\varepsilon Y$. That the Ontology Axiom implies (ATB) follows similarly.

We now introduce a constant name AT.

Definition 40. $X\varepsilon AT \iff X\varepsilon X \wedge \forall P,Q.(P < X \wedge Q < X \implies P < Q)$.

The term $X\varepsilon AT$ reads "X is an atom".
We may check that

Proposition 41. *(ATB) is an axiom for atomic boolean algebra without the zero element.*

The proof consists in a straightforward checking of equivalence.
We have therefore

Proposition 42. *[cf. [37]] Theorems of ontology are those which are true in every model for an atomic boolean algebra without the null element.*

3.6 Ontology in Information Systems

Let us observe that in an information system the Leibnizian principle of indiscernibility is observed in the following form:

$$a(x) = a(y) \text{ for each } a \in A \textbf{ if and only if } [x]_A = [y]_A.$$

It follows that we have three types of entities when considering an information system A: objects (corresponding to rows in the information table), indiscernibility classes of objects (over various distinct subsets of attributes), as well as collections of entities of two former types.

We propose to discuss here two types of Ontology, related to each other, but distinct in the light of rough mereology (see Section 5.2). We discuss them below.

Ontology of B–indiscernibility We form first individual entities. Recall that the symbol $[u]_B$ denotes the indiscernibility class of u over B. We call a *B–individual* any B–exact subset of the universe U. Then, we introduce *B–names* as sets of B–individuals. Then, $X \varepsilon Y$ means that X is a B–individual and Y is a set of such individuals cntaining X.

Let us observe that some (regular) names may be introduced via templates e.g. given $C \subset B$ and $v = Inf_C(u)$ for some $U \in U$, we may interpret the template (C, v) in a twofold way.

1. We may take the meaning $[C, v]$ of the template (C, v) as denoting the *individual* i.e. the set $\{u \in U : \forall a \in C.a(u) = v_a\}$;
2. We may take (C, v) as a name i.e. the set of all individuals $[B, w]$ such that $w|C = v$.

Ontology of A–indiscernibility Our next Ontology will formally be an extension of the former one. We define individual entities as A–exact sets. General names will be defined as sets (collections, lists) of individual entities. Again, we may use templates to express certain (regular) names :

1. We may take the meaning $[C, v]$ of the template (C, v) as denoting the A–*individual* i.e. the set $\{u \in U : \forall a \in C.a(u) = v_a\}$;
2. We may take (C, v) as a name i.e. the set of all individuals $[D, w]$ such that $C \subseteq D \wedge w|C = v$.

We may introduce propositional connectives $\vee, \wedge$ into templates to form the *template logic* corresponding to the boolean structure in Calculus of Names viz.

1. $(C, v) + (D, w)$ is the name containing those individuals which either fall in (C, v) or they fall in (D, w);
2. $(C, v) \cdot (D, w)$ is the name containing those individuals which fall both in (C, v) and (D, w);

3. $(C, v) - (D, w)$ is the name containing those individuals which fall in (C, v) but not in (D, w).

One may see that a general name for A–exact sets may be written down in the form $+_{i=1}^{k}(C_i, v_i)$ for an appropriate k and a choice of $C_i \subseteq U, v_i = Inf_{C_i}(u_i), u_i \in U$ where $i = 1, 2, .., k$. Thus, we have for instance: $X \varepsilon Y$ when $Y =_E +_{i=1}^{k}(C_i, v_i) \wedge \exists i.X = [C_i, v_i]$. Similar considerations are valid for the case of B–exact sets.

We return to these examples in Sections on Mereology and Rough Mereology.

4 The Leśniewski Mereology

Mereology is a theory of collective classes i.e. individual entities representing general names as opposed to Ontology which is a theory of distributive classes i.e. general names. Mereology may be based on each of a few notions like those of a *part*, an *element*, a *class* etc. Historically it has been conceived by Stanisław Leśniewski [51], [53], [54] cf. [20], [89], [83], [84], as a theory of the relation *part* and we here follow this line of development. In particular, we present the development of Mereology within Ontology. The meaning of the copula ε explained above, we may now make use of this notation in what follows.

We assume that the copula ε is given and that the Ontology Axiom holds. Under these assumptions, we introduce the notion of the name-forming functor *pt* of *part*. Our presentation is based on [53] in the first place.

4.1 Mereology axioms

(A1) $X \varepsilon pt(Y) \Longrightarrow X \varepsilon X \wedge Y \varepsilon Y$;
this means that the functor *pt* is defined for individual entities only.
(A2) $X \varepsilon pt(Y) \wedge Y \varepsilon pt(Z) \Longrightarrow X \varepsilon pt(Z)$;
meaning that the functor *pt* is transitive i.e. a part of a part is a part.
(A3) $\text{non}(X \varepsilon pt(X))$;
which means that the functor *pt* is non–reflexive (or, equivalently, if $X \varepsilon pt(Y)$ then $non(Y \varepsilon pt(X)$.

On the basis of the notion of part, we define the notion of an *element* (an improper, possibly, part) as a name- forming functor *el*. (We note in passing that in the original scheme of Leśniewski this notion had the name of an *ingredient*).

Definition 43. $X \varepsilon el(Y) \Longleftrightarrow X \varepsilon pt(Y) \vee X = Y$

It is clearly possible to introduce mereology in terms of the functor *el* as a partial order functor (i.e. being, consecutively, *reflexive*: $X \varepsilon el(X)$, *transitive* : $X \varepsilon el(Y) \wedge Y \varepsilon el(Z) \Longrightarrow X \varepsilon el(Z)$, *weakly symmetric*: $X \varepsilon el(Y) \wedge Y \varepsilon el(X) \Longrightarrow X = Y$) cf. Proposition 45.

The remaining axioms of mereology are related to the class functor which converts distributive classes (general names) into individual entities. The class operator Kl is a principal tool in applications of Rough Mereology to problems of Distributed Systems, Knowledge Granulation, Computing with Words where it does play the role of granulating (clustering) operator allowing for forming granules of knowledge and subsequently instrumental in calculi on them cf. [64], [65], [66], [67], [75].

We may now introduce the notion of a (collective) class via a name-forming functor Kl.

Definition 44. $X\varepsilon Kl(Y) \iff$
$$\exists Z.Z\varepsilon Y \wedge \forall Z.(Z\varepsilon Y \Longrightarrow Z\varepsilon el(X)) \wedge \forall Z.(Z\varepsilon el(X) \Longrightarrow \exists U,W.(U\varepsilon Y \wedge W\varepsilon el(U) \wedge W\varepsilon el(Z)).$$

Let us disentangle the meaning of this Definition. First, we may realize that the class operator Kl is intended as the operator converting names (general sets of entities) into individual entities i.e. collective classes; its role may be fully compared to the role of the union of sets operator in the classical set theory. The analogy is indeed not only functional but also formal.

Let us look at the subsequent conjuncts in the defining formula above.

1. $\exists Z.Z\varepsilon Y$;
   this means that Y is a non–empty name (recall that the union of the empty family of sets is the empty set hence prohibited in Ontology).
2. $\forall Z.(Z\varepsilon Y \Longrightarrow Z\varepsilon el(X))$;
   meaning that any individual listed in Y is an element of $Kl(Y)$ (compare with: any element of the family of sets is a subset of the union of that family).
3. $\forall Z.(Z\varepsilon el(X) \Longrightarrow \exists U,W.(U\varepsilon Y \wedge W\varepsilon el(U) \wedge W\varepsilon el(Z))$;
   this means that any element of $Kl(Y)$ has an element in common with an individual in Y (similarly, any element in the union of a family of sets is an element in at least one member of this family).

Thus, the class functor pastes together individuals in Y by means of their common elements

The class functor is subject to the following postulates.

(A4) $X\varepsilon Kl(Y) \wedge Z\varepsilon Kl(Y) \Longrightarrow X\varepsilon Z$;
this means that $Kl(Y)$ is an individual name (entity), for any (non–empty) Y.
(A5) $\exists Z.Z\varepsilon Y \iff \exists Z.Z\varepsilon Kl(Y)$;
meaning that $Kl(Y)$ exists (i.e. is a non–empty individual name) if and only if Y is a non–empty name.

4.2 First Consequences

From axioms above, we start a build-up of mereology. We begin with simple consequences of axioms.

Proposition 45. *1.* $X\varepsilon el(Y) \wedge Y\varepsilon el(Z) \Longrightarrow X\varepsilon el(Z)$;
2. $X\varepsilon el(Y) \wedge Y\varepsilon el(X) \Longrightarrow X = Y$;

3. $X\varepsilon el(X)$.

Proposition 46. $X\varepsilon X \Longrightarrow X\varepsilon Kl(elX)$ *where* $Z\varepsilon elX \Longleftrightarrow Z\varepsilon el(X)$.

Proof. Assume $X\varepsilon X$; hence $X\varepsilon el(X)$ and so $\exists Z.Z\varepsilon el(X)$. For each $Z\varepsilon el(X)$ the formula $\exists U,W.\ U\varepsilon el(X) \wedge W\varepsilon el(U) \wedge W\varepsilon el(Z))$ is satisfied with $U = W = Z$ and so $X\varepsilon Kl(elX)$.

Proposition 47. $X\varepsilon el(Y) \Longleftrightarrow \exists Z.X\varepsilon Z \wedge Y\varepsilon Kl(Z)$

Proof. $X\varepsilon el(Y)$ implies by Proposition 46 that $Z = Kl(Y)$ satisfies the right hand formula; the converse follows from the class definition.

We define new name-forming functors.

Definition 48. 1. $X\varepsilon partY \Longleftrightarrow X\varepsilon pt(Y)$;
2. $X\varepsilon(\varepsilon Y) \Longleftrightarrow X\varepsilon Y$.

We have counterparts of Proposition 47.

Proposition 49. $X\varepsilon X \Longrightarrow X\varepsilon Kl(\varepsilon X)$.

Proof. Assume $X\varepsilon X$; then $\exists Z.Z\varepsilon X, \forall Z.(Z\varepsilon X \Longrightarrow Z\varepsilon el(X))$,
$\forall Z.(Z\varepsilon el(X) \Longrightarrow \exists U,W.(U\varepsilon X \wedge W\varepsilon el(U) \wedge W\varepsilon el(Z))$ are satisfied.

Proposition 50. $\exists Z.Z\varepsilon pt(X) \Longrightarrow X\varepsilon Kl(partX)$.

Proof. Assume $\exists Z.Z\varepsilon pt(X)$; it suffices to check that $T\varepsilon el(X) \Longrightarrow$
$\exists U,W.W\varepsilon el(U) \wedge W\varepsilon el(T) \wedge U\varepsilon pt(X)$; in case $T = X$, let $U = W = Z$ where $Z\varepsilon pt(X)$ and in case $T\varepsilon pt(X)$ let $U = W = T$.

Proposition 51. $X\varepsilon X \Longrightarrow X\varepsilon Kl(X)$.

We now define the notion of a set , weakr than that of a class; we may observe that class is the set with the universality property: $\forall Z.(Z\varepsilon Y \Longrightarrow Z\varepsilon el(Kl(Y))$.

Definition 52. $X\varepsilon set(Y) \Longleftrightarrow \exists Z.Z\varepsilon Y \wedge \forall Z.(Z\varepsilon el(X) \Longrightarrow$
$\exists U,W.(U\varepsilon el(Z) \wedge U\varepsilon Y \wedge W\varepsilon el(U) \wedge W\varepsilon el(Z))$.

We now recall following [51], [53] some technical propositions leading to thesis (Proposition 61) equivalent to (A4) and giving an inference rule about the functor el.

Proposition 53. $X\varepsilon set(Y) \wedge \forall Z.(Z\varepsilon Y \Longrightarrow Z\varepsilon W) \wedge T\varepsilon el(X) \Longrightarrow$
$\exists P,R.(P\varepsilon el(T) \wedge P\varepsilon el(R) \wedge R\varepsilon W \wedge R\varepsilon el(X)$.

Proof. From $X\varepsilon set(Y)$ and $T\varepsilon el(X)$ it follows that $A\varepsilon el(T)$, $A\varepsilon el(B)$, $B\varepsilon el(X)$, $B\varepsilon Y$ hence $\forall Z.(Z\varepsilon Y \Longrightarrow Z\varepsilon W)$ implies $B\varepsilon W$ for some A, B so A, B satisfy the consequent.

Proposition 54. $X\varepsilon set(Y) \wedge \forall Z.(Z\varepsilon Y \Longrightarrow Z\varepsilon W) \Longrightarrow X\varepsilon set(W)$.

Proposition 55. $X\varepsilon Y \Longrightarrow X\varepsilon set(Y)$.

Proposition 56. $X\varepsilon Kl(Y) \Longrightarrow X\varepsilon set(Y)$.

Proofs are obvious.

Proposition 57. $X\varepsilon Kl(set(Y)) \wedge Z\varepsilon el(X) \Longrightarrow$
$\exists U, W.(U\varepsilon el(Z) \wedge U\varepsilon el(W) \wedge W\varepsilon Y \wedge W\varepsilon el(X))$.

Proof. Assume that $X\varepsilon Kl(set(Y)), Z\varepsilon el(X)$;there exist U, W with the properties $U\varepsilon el(Z), U\varepsilon el(W), W\varepsilon set(Y), W\varepsilon el(X)$ hence there exist P, Q with $P\varepsilon el(U)$, $P\varepsilon el(Q)$, $Q\varepsilon el(W)$, $Q\varepsilon Y$ implying $Q\varepsilon el(X)$. Thus P, Q satisfy the consequent.

Corollary 58. $X\varepsilon Kl(set(Y)) \Longrightarrow X\varepsilon Kl(Y)$.

Proof. Assume that $X\varepsilon Kl(set(Y))$;then, $Z\varepsilon Y$ implies $Z\varepsilon set(Y)$ by Proposition 55 and finally, by the assumption, $Z\varepsilon el(X)$. Now, for $Z\varepsilon el(X)$,there exist U, W with $U\varepsilon el(Z)$, $U\varepsilon el(W)$, $W\varepsilon set(Y)$,so by the definition of a set (Definition 52), we may assume that $\exists P, Q.P\varepsilon el(U)$, $P\varepsilon el(Q)$, $Q\varepsilon Y$. It follows that $X\varepsilon Kl(Y)$.

Corollary 59. $X\varepsilon Kl(set(Y)) \Longleftrightarrow X\varepsilon Kl(Y)$.

Sets are elements of classes.

Proposition 60. $X\varepsilon set(Y) \Longrightarrow X\varepsilon el(Kl(Y))$.

Proof. $X\varepsilon set(Y)$ implies $\exists U.U\varepsilon Y$ hence $\exists Z.Z\varepsilon Kl(Y)$ by (A5) and $Z\varepsilon Kl(set(Y))$ by Corollary 59 so finally $X\varepsilon el(Z)$.

Proposition 61. $X\varepsilon X \wedge \forall Z.(Z\varepsilon el(X) \Longrightarrow \exists T, T\varepsilon el(Z) \wedge T\varepsilon el(Y)) \Longrightarrow X\varepsilon el(Y)$.

Actually, one may prove that this proposition is equivalent to (A4). It may be regarded as an inference rule about the functor *el* and also as an alternative axiom.

Proof. From $X\varepsilon X, \forall Z.(Z\varepsilon el(X) \Longrightarrow \exists T.T\varepsilon el(Z) \wedge T\varepsilon el(Y))$ it follows that X ε $set(el(Y))$ and by Proposition 60, $X\varepsilon el(Kl(el(Y))$ hence by Proposition 46, $X\varepsilon el(Y)$.

4.3 Subset, Complement, Relations

We define the notions of a subset, a complement and we will look at relations and functions in mereological context.We define first the notion of a subset as a name-forming functor *sub* of an individual variable.

Definition 62. $X \varepsilon sub(Y) \iff X \varepsilon X \wedge Y \varepsilon Y \wedge \forall Z (Z \varepsilon el(X) \implies Z \varepsilon el(Y))$.

Proposition 63. $X \varepsilon sub(Y) \implies X \varepsilon el(Y)$.

Proof. As $X \varepsilon el(X)$ by Proposition 45, it follows that $X \varepsilon sub(Y)$ implies $X \varepsilon el(Y)$.

Proposition 64. $X \varepsilon el(Y) \implies X \varepsilon sub(Y)$.

Corollary 65. $X \varepsilon el(Y) \iff X \varepsilon sub(Y)$.

We now define the notion of being external as a binary proposition–forming functor *ext* of individual variables.

Definition 66. $ext(X,Y) \iff X \varepsilon X \wedge Y \varepsilon Y \wedge non(\exists Z . Z \varepsilon el(X) \wedge Z \varepsilon el(Y)$.

Proposition 67. *1.* $X \varepsilon X \implies non(ext(X,X))$;
2. $ext(X,Y) \iff ext(Y,X)$.

The notion of a complement is rendered as a name-forming functor *comp* of two individual variables. We first define a new name Θ as follows: $U \varepsilon \Theta \iff U \varepsilon el(Z) \wedge ext(U,Y)$.

Definition 68. $X \varepsilon comp(Y,Z) \iff Y \varepsilon sub(Z) \wedge X \varepsilon Kl(\Theta)$.

Proposition 69. $X \varepsilon comp(Y,Z) \implies ext(X,Y)$.

Proof. As $X \varepsilon comp(Y,Z)$, if $T \varepsilon el(X)$ then we have U,W with $U \varepsilon el(T)$, $U \varepsilon el(W)$, $W \varepsilon el(Z)$, $ext(W,Y)$ hence $non(U \varepsilon el(Y))$ and thus $non(U \varepsilon el(Y))$.

Corollary 70. $X \varepsilon X \implies non(X \varepsilon comp(X,Z))$.

Proposition 71. $X \varepsilon comp(Y,Z) \implies X \varepsilon el(Z)$

Proof. As $X \varepsilon comp(Y,Z)$ implies $X \varepsilon Kl(\Theta)$ hence if $T \varepsilon el(X)$ then we have U,W with $U \varepsilon el(T)$, $U \varepsilon el(W)$, $W \varepsilon el(Z)$ hence by Proposition 61, $X \varepsilon el(Z)$.

Coming to relations, already defined in Ontology, we will introduce a Proposition–forming functor $\longrightarrow (\varphi, X, Y)$ where X, Y are individual names and φ is a relation.

Definition 72. $\longrightarrow(\varphi, X, Y)$ iff $X\varepsilon X \wedge Y\varepsilon Y \wedge \forall T.(T\varepsilon el(X) \wedge \varphi(T,U) \Longrightarrow U\varepsilon el(Y))$.

Proposition 73. *1.* $\longrightarrow(\varphi, X, Y) \wedge Z\varepsilon el(X) \Longrightarrow\longrightarrow(\varphi, Z, Y)$;
2. $\longrightarrow(\varphi, X, Z) \wedge Z\varepsilon el(Y) \Longrightarrow\longrightarrow(\varphi, X, Y)$;
3. $\longrightarrow(\varphi, X, Y) \Longrightarrow\longrightarrow(\varphi, X, Kl(el(Y))$;
4. $\longrightarrow(\varphi, X, Y) \Longrightarrow\longrightarrow(\varphi, Kl(el(X), Y)$.

Proof. Obvious.

4.4 Other Axiomatics, Completeness

As with Ontology, Mereology may be axiomatized in terms of notions derived from the *part* functor. We begin with an axiom due to Sobociński [83], which formalizes mereology in terms of the functor *el*.

The Sobociński Axiom

(S) $X\varepsilon el(Y) \Longleftrightarrow Y\varepsilon Y \wedge \forall f, Z.[\forall C.(C\varepsilon f(Z) \Longleftrightarrow$
$(\forall D.D\varepsilon Z \Longrightarrow D\varepsilon el(C)) \wedge (\forall D.D\varepsilon el(C) \Longrightarrow \exists E, F.E\varepsilon Z \wedge F\varepsilon el(D) \wedge F\varepsilon el(E))$
$\wedge Y\varepsilon el(Y) \wedge Y\varepsilon Z] \Longrightarrow X\varepsilon el(f(Z))$.

It is not difficult to see that (S) is a theorem of mereology.

Proposition 74. *(S) is a thesis of mereology*

Proof. It is easily seen that $f(Z)$ is $Kl(Z)$ so (S) reads as the thesis: $X\varepsilon el(Y) \Longleftrightarrow Y\varepsilon Y \wedge \forall Z.(Y\varepsilon Z \Longrightarrow X\varepsilon el(Kl(Z))$ which is true in Mereology.

Proposition 75. *(S) implies axioms of mereology*

Proof. Assume (S); then $\exists Z.Z\varepsilon el(Y)$ $(the\ left\ side)$ implies $Y\varepsilon el(Y)$. Similarly, taking into account that $f(Z)$ denotes $Kl(Z)$, we find that $X\varepsilon Kl(elX)$ and from this we obtain that $X\varepsilon el(Y) \wedge Y\varepsilon el(Z) \Longrightarrow X\varepsilon el(Z)$. Letting $X\varepsilon pt(Y) \Longleftrightarrow X\varepsilon el(Y) \wedge non(X = Y)$, we arrive at (A1) and (A2). The uniqueness of $Kl(Z)$ follows from the subformula $C\varepsilon f(Z) \Longleftrightarrow$
$(\forall D.D\varepsilon Z \Longrightarrow D\varepsilon el(C)) \wedge (\forall D.D\varepsilon el(C) \Longrightarrow \exists E, F.E\varepsilon Z \wedge F\varepsilon el(D) \wedge F\varepsilon el(E)$
of (S) and similarly, it does imply that $\exists E.E\varepsilon Z \Longleftrightarrow \exists C.C\varepsilon Kl(Z)$.

A similar axiom has been proposed by Lejewski [50].

The Lejewski Axiom

(L) $X\varepsilon el(Y) \Longleftrightarrow Y\varepsilon Y \wedge \forall f, Z, C.\{\forall D.[D\varepsilon f(Z) \Longleftrightarrow$
$(\forall E.(\exists F.F\varepsilon el(D) \wedge F\varepsilon el(F)) \Longleftrightarrow \exists G, H.G\varepsilon Z \wedge H\varepsilon el(E) \wedge H\varepsilon el(G))]$
$\wedge Y\varepsilon el(Y) \wedge Y\varepsilon el(C) \wedge Y\varepsilon Z]\} \Longrightarrow X\varepsilon el(f(Z))$.

One can show similarly that (L) and (S) are equivalent.

A paraphrase of these axioms has been proposed by Clay [15].
The Clay Axiom

(Cl) $X\varepsilon el(Y) \Longleftrightarrow \{X\varepsilon X \wedge Y\varepsilon Y \wedge Y\varepsilon el(Y) \Longrightarrow \forall U,W.(Y\varepsilon U\wedge$
$(\forall C.(C\varepsilon W \Longleftrightarrow \forall D.(D\varepsilon U \Longrightarrow D\varepsilon el(C)) \wedge \forall D.(D\varepsilon el(C) \Longrightarrow$
$\exists E,F.(E\varepsilon U \wedge F\varepsilon el(D) \wedge F\varepsilon el(E)) \Longrightarrow X\varepsilon el(W)\}.$

We may formally replace the functor *el* with a new name $\leq$ and we may read (and write down) $X\varepsilon X$ as $X\varepsilon V$; replacing U,W by small letters u,w symbolizing not necessarily individual names and denoting by f the field of $\leq$, we arrive at (Cl) in the for

(M) $X\varepsilon \leq (Y) \Longleftrightarrow \{X\varepsilon V \wedge Y\varepsilon V \wedge Y\varepsilon \leq (Y) \Longrightarrow \forall u,w.(u \subset f \wedge w \subset f \wedge Y\varepsilon u\wedge$
$(\forall C.(C\varepsilon w \Longleftrightarrow \forall D.(D\varepsilon u \Longrightarrow D\varepsilon \leq \ (C)) \wedge \forall D.(D\varepsilon \leq (C) \Longrightarrow$
$\exists E,F.(E\varepsilon u \wedge F\varepsilon \leq (D) \wedge F\varepsilon \leq (D)) \Longrightarrow X\varepsilon \leq (w)\}.$

Substituting for V the universe U, neglecting the copula and taking into account that w must be an individual name as pointed to by (M), one gets following [15],

(CBA) $X \leq Y \Longleftrightarrow \{X \in U\wedge Y \in U\wedge Y \leq Y \Longrightarrow \forall u,w.(u \subset U\wedge w \subset U\wedge Y \in u\wedge$
$(\forall C.(C \in w \Longleftrightarrow \forall D.(D \in u \Longrightarrow D \leq C) \wedge \forall D.(D \leq C \Longrightarrow$
$\exists E,F.(E \in u \wedge F \leq D \wedge F \leq E)) \Longrightarrow \exists L.\ w = \{L\} \wedge X \leq L.$

It may be checked that (CBA) is the axiom for a complete boolean algebra without the null element.

Therefore

Proposition 76. *[Tarski [91]]*
Models of mereology are models of complete boolean algebras without zero.

Mereology is therefore complete with respect to algebraic structures which are models for complete boolean algebras without zero.

4.5 Mereology in Information Systems

As with Ontology, we discern between two basic types of mereological structures. We refer to two kinds of Ontology (cf.Section 3.6)

Mereology of B–indiscernibility Recall that individual entities in this case are B–exact sets and general names are collections (sets, lists) of B–exact sets. Recalling our usage of templates, we may write down any general name as $+_{i=1}^{k}(B,v_i)$ where $v_i = Inf_B(u_i)$ for some $u_i \in U$ while $[B,v]$ denotes the B–indiscernibility class $[u]_B$ where $v = Inf_B(u)$.

We define the functor *pt* in this case: for an individual $Y = \bigcup_{i=1}^{k}[B,v_i]$, we let $X\varepsilon pt(Y) \Longleftrightarrow k \geq 2 \wedge \exists i.X = [B,v_i]$.

Thus, we propose that parts of which B–exact sets are formed be simply B–indiscernibility classes; clearly, a B–indiscernibility class has no parts itself–it is an *atom*.

Let us observe that : given a (regular) name $Y = +_{i=1}^{k}(B, v_i)$, we have: $Kl(Y) = \bigcup_{i=1}^{k}[B, v_i]$ i.e. the class of Y is the individual $X = \bigcup_{i=1}^{k}[B, v_i]$; indeed, by the class definition, it sufices to check that any part of Y i.e. any $Z_i = [B, v_i]$ has an element (e.g. Z_i itself) which is in turn an element of an individual (again, Z_i itself) which is in Y (and obviously, $Z_i \varepsilon Y$).

In particular, we have: $Kl((B, v)) = [B, v]$ for any (B, v).

Mereology of A–indiscernibility: all B–indiscernibility classes are atoms We may just repeat the considerations in the preceding paragraph. We define the functor *pt* in this case: for an individual $Y = \bigcup_{i=1}^{k}[C_i, v_i]$, we let $X\varepsilon pt(Y) \iff k \geq 2 \wedge \exists i.X = [C_i, v_i]$.

Thus, we propose that parts of which A–exact sets are formed are simply A–indiscernibility classes; clearly, an A–indiscernibility class has no parts itself–it is an *atom*.

Let us observe that : given a (regular) name $Y = +_{i=1}^{k}(C_i, v_i)$, we have: $Kl(Y) = \bigcup_{i=1}^{k}[C_i, v_i]$ i.e. the class of Y is the individual $X = \bigcup_{i=1}^{k}[C_i, v_i]$; indeed, by the class definition, it sufices to check that any part of Y i.e. any $Z_i = [C_i, v_i]$ has an element (e.g. Z_i itself) which is in turn an element of an individual (again, Z_i itself) which is in Y (and obviously, $Z_i \varepsilon Y$).

Mereology of A–indiscernibility: only A–indiscernibility classes are atoms We may modify the definition of a part given above: for an individual name $Y = \bigcup_{i=1}^{k}[C_i, v_i]$, we let $X\varepsilon pt(Y) \iff k \geq 2 \wedge \exists i.X = [C_i, v_i] \vee X = [D, w] \wedge \exists i.C_i \subset D \wedge w|c_i = v_i$; it follows immediately, that atoms in this case are only A–indiscernibility classes.

Again, we have here that: given a (regular) name $Y = +_{i=1}^{k}(C_i, v_i)$, we have: $Kl(Y) = \bigcup_{i=1}^{k}[C_i, v_i]$ i.e. the class of Y is the individual $X = \bigcup_{i=1}^{k}[C_i, v_i]$.

The above examples explain the role of the class operator in information systems; we may exploit mereology also in the task of creating Ontology of non–exact concepts (names). Here, we find usage for the mereological functor of a subset.

Ontology of inexact (rough) concepts We define a relation *rough–approx* by letting *rough–approx*$(X, Y) \iff X\varepsilon X \wedge Y\varepsilon Y \wedge X\varepsilon sub(Y)$. We give the relation *rough–approx* a new name via $(X, Y)\varepsilon\ ROUGH \iff$*rough–approx*$(X, Y)$.

Then $ROUGH$ is the name which contains all pairs approximating inexact concepts from below as well as from above (possibly, with some surplus: some pairs (X, Y) may not define an inexact concept).

5 Rough Mereology

Rough mereology is an extension of mereology based on the predicate of being a part in a degree; this predicate is rendered here as a family of name–forming functors μ_r parameterized by a real parameter r in the interval $[0,1]$ with the intent that $X\varepsilon\mu_r(Y)$ reads "*X is a part of Y in degree at least r*". We begin with the set of axioms and we construct the axiom system as an extension of systems for ontology and mereology.

We assume thus that a functor el of an element satisfying the mereology axiom system is given; around this, we develop a system of axioms for rough mereology.

5.1 The axiom system

The following is the list of basic postulates.

(RM1) $X\varepsilon\mu_1(Y) \Longleftrightarrow X\varepsilon el(Y)$;
this means that being a part in degree 1 is equivalent to being an element: this establishes the connection between rough mereology and mereology.
(RM2) $X\varepsilon\mu_1(Y) \Longrightarrow \forall Z.(Z\varepsilon\mu_r(X) \Longrightarrow Z\varepsilon\mu_r(Y))$;
meaning the monotonicity property: any object Z is a part of Y in degree not smaller than that of being a part in x whenever X is an element of Y.
(RM3) $X = Y \wedge X\varepsilon\mu_r(Z) \Longrightarrow Y\varepsilon\mu_r(Z))$;
this means that the identity of individuals is a congruence with respect to μ.
(RM4) $X\varepsilon\mu_r(Y) \wedge s \leq r \Longrightarrow X\varepsilon\mu_s(Y)$;
establishes the meaning " a part in degree at least r".

It follows that the functor μ_1 coincides with the given functor el establishing a link between rough mereology and mereology while functors μ_r with $r < 1$ diffuse the functor el to a hierarchy of functors expressing being an element (or, part) in various degrees.

We introduce a new name-forming functor μ_r^+ via

Definition 77. $X\varepsilon\mu_r^+(Y) \Longleftrightarrow X\varepsilon\mu_r(Y) \wedge \forall s.(s > r \Longrightarrow non(X\varepsilon\mu_s(Y))$.

We find a usage for this functor in two new postulates.

(RM5) $ext(X,Y) \Longrightarrow X\varepsilon\mu_0^+(Y)$.

This postulate expresses our intuition that objects which are external to each other should be elements of each other in no positive degree. This assumption however reflects a high degree of certainty of our knowledge and it will lead to models in which connection coincides with overlapping (see below). It will be more realistic to assume that our knowledge is uncertain to the extent that we may not be able to state beyond doubt that two given objects are external to each other, rather we will be pleased with the atatement that they are in such case elements of each other in a bounded degree. Hence, we introduce a weaker form of the postulate (RM5).

(RM5*) $ext(X,Y) \Longrightarrow \exists s < 1.X\varepsilon\mu_s^+(Y)$.

5.2 Rough Mereology in Information Systems

We will refer to Section 4.5, where examples of mereological decompositions into parts were given. We will propose here a few measures of partness based on either individual frequency count or on attribute–value frequency count.

Rough membership functions Here, we apply the idea of a rough membership function of Pawlak and Skowron [63]; we recall that given a subset (in the set–theoretic sense) X of the universe U of an information system A=(U, A), one defines the rough membership function μ_X by letting $\mu_X(u) = \frac{card(X \cap [u]_A)}{card([u]_A)}$ for $u \in U$.

This notion may be extended [64] to a notion of a rough membership function defined on concepts (i.e. subsets of U): given two non–empty concepts X, Y, we let: $\mu_X(Y) = \frac{card(X \cap Y)}{card(Y)}$. Thus, $\mu_X(Y)$ is a measure of the degree in which Y is contained in X.

Rough Mereology of B–exact sets: the row frequency count We apply the rough membership function in its generalized form. Given two B–exact sets, X, Y, we let $r = \mu_X(Y)$ and accordingly $Y \varepsilon \mu_r(X)$. This measure μ is thus based on the frequency count of rows of the information table in respectively, X and Y. From probabilistic point of view, it may be regarded as an unbiased estimate of the conditional probability $Pr(X|Y)$ cf. [62].

Rough Mereology of B–exact sets: the B–class frequency count Here, we propose yet another measure based on the rough membership function; we apply it counting this time the number of B–indiscernibility classes in respectively $X \cap Y$ and Y. Accordingly, for $X = \bigcup_{i=1}^{k} [B, v_i]$ and $Y = \bigcup_{j=1}^{m} [B, w_j]$, we let :$r = \frac{card(\{[B,v_i]: i \leq k\} \cap \{[B,w_j]: j \leq m\})}{m}$ and accordingly, $Y \varepsilon \mu_r(X)$.

Rough Mereology of A–exact sets : the case when all indiscernibility classes are atoms We may apply here either of the two frequency count measures defined earlier: in the first case, for two A–exact sets $X = \bigcup_{i=1}^{k} [B_i, v_i]$ and $Y = \bigcup_{j=1}^{m} [C_j, w_j]$, we let: $r = \frac{card(X \cap Y)}{card(Y)}$ and accordingly, $Y \varepsilon \mu_r(X)$.

In the second case, we let: $r = \frac{card(\{[B_i,v_i]: i \leq k\} \cap \{[C_j,w_j]: j \leq m\})}{m}$ and accordingly, $Y \varepsilon \mu_r(X)$.

We now propose a method for extending a measure defined for elements of two concepts to a measure on these two concepts; this idea will be applied in the following section.

Extending rough inclusions Assume that we are given two individuals X, Y being classes of (finite) names: $X = Kl(X')$, $Y = Kl(Y')$ and that we have defined values of μ for pairs T, Z of individuals where $T\varepsilon X'$, $Z\varepsilon Y'$.

We extend μ to a measure μ^* on X, Y by letting:

$r = min_{Z\in Y'}\{max_{T\in X'} max\{s : Z\varepsilon\mu_s(T)\}\}$ and $Y\varepsilon\mu^*_r(X)$.

It may be proved straightforwardly that

Proposition 78. *The measure μ^* satisfies (RM1)–(RM4).*

Rough Mereology of A–exact sets : the case when atoms are A–indiscernibility classes We apply the idea of the last section to this case. First, we define μ on elementary individuals: given $X = [B, v]$ and $Y = [C, w]$, we define the set $IND(X, Y) = \{a \in A : a \in B \cap C \wedge v_a = w_a\}$ and then we let: $r = \frac{card(IND(X,Y))}{card(B)}$ and finally $Y\varepsilon\mu_r(X)$. Thus, the degre of partness of T in X is determined by frequency count of identical elementary descriptors in templates (B, v) and (C, w).

Now, given individual entities $X = \bigcup_{i=1}^{k}[B_i, v_i]$ and $Y = \bigcup_{j=1}^{m}[C_j, w_j]$, we let $r = min_{[C_j,w_j]}\{max_{[B_i,v_i]} max\{s : [C_j, w_j]\varepsilon\mu_s([B_i, v_i])\}\}$ and $X\varepsilon\mu^*_r(X)$.

Example 1. We give a simple example concerning the last method of calculating the measure μ. We begin with an example of an information table.

	a_1	a_2	a_3
u_1	1	0	1
u_2	1	0	0
u_3	1	1	0
u_4	0	1	1
u_5	0	1	0
u_6	1	0	1
u_7	1	1	0

Table 1. *Binary1*: An example of an information table

Consider $B = \{a_1, a_2\}$, $C = \{a_2, a_3\}$, $v =< 1, 0 >$, $w =< 0, 1 >$; for $X = [B, v]$, $Y = [C, w]$, we have $IND(X, Y) = \{a_2\}$ and accordingly, $Y\varepsilon\mu^*_{0.5}(X)$.

We now have at our disposal some recipes for introducing rough inclusions in information systems. The choice may depend on the context; let us observe that we may also have also some parameterized formulae, subject to optimization in a given context.

Rough Mereology in Information systems: measures induced from rows The following procedure may define a rough inclusion in an information system (U, A) in case we would like to start with rows of the information table.

1. Consider a partition $P = \{A_1, A_2, ..., A_k\}$ of A.
2. Select a convex family of coefficients: $W = \{w_1, w_2, ..., w_k\}$.
3. Define $IND(A_i)(u, u') = \{a \in A_i : a(u) = a(u')\}$.
4. Let $r = \sum_{i=1}^{k} w_i \frac{card(IND(A_i)(u,u'))}{card(A_i)}$.
5. Let $u\varepsilon\mu_r(u')$.

In this way we may relate rows one to another. Observe that this rough inclusion is symmetric: $u\varepsilon\mu_r(u') \Longleftrightarrow u'\varepsilon\mu_r(u)$.

Now, we may extend this measure to elementary individuals and then to individuals as indicated in Example. For instance, taking $w_1 = 1$, we have $u_1\varepsilon\mu_{0.66(6)}u_2$. Applying the extension μ^*, we have for X, Y in Example 1 above: $X = \{u_1, u_2, u_6\}$, $Y = \{u_2, u_6\}$ and $Y\varepsilon\mu_1^*(X)$ (in this model, elementary individuals (i.e. atoms) are rows of the information table hence $Y\varepsilon el(X)$).

5.3 Renormalization: t-norm modifiers

We introduce now, following Polkowski& Skowron [64], a modification to our functors μ_r; it is based on an application of residuated implication [41] and a measure of containment defined within the fuzzy set theory (the necessity measure) [36], [7]. Combining the two ideas, we achieve a formula for μ_r which allows for a transitivity rule; this rule will in turn allow to introduce into our universe rough mereological topologies.

We therefore recall the notion of a $t-norm$ $\top$ as a function of two arguments $\top : [0,1]^2 \longrightarrow [0,1]$ which satisfies the following requirements:

1. $\top(x, y) = \top(y, x)$;
2. $\top(x, \top(y, z)) = \top(\top(x, y), z)$;
3. $\top(x, 1) = x$;
4. $x' \geq x \wedge y' \geq y \Longrightarrow \top(x', y') \geq \top(x, y)$.

We also invoke a notion of fuzzy containment $\subset_r$ based on necessity cf. [36]; it relies on a many-valued implication Υ i.e. on a function $\Upsilon : [0,1]^2 \longrightarrow [0,1]$ according to the formula:

$$X \subset_r Y \Longleftrightarrow \forall Z.(\Upsilon(\mu_X(Z), \mu_Y(Z)) \geq r)$$

where μ_A is the fuzzy membership function [41] of the fuzzy set A.

We replace Υ with a specific implication viz. the residuated implication $\overrightarrow{\top}$ induced by $\top$ and defined by the following prescription.

Definition 79. $\overrightarrow{\top}(r, s) \geq t \Longleftrightarrow \top(t, r) \leq s$.

We define a functor $\mu_{\top,r}$ where $r \in [0,1]$, according to the recipe

Definition 80. $X\varepsilon\mu_{\top,r}(Y) \iff X\varepsilon X \wedge \forall Z.(\exists t,w.Z\varepsilon\mu_t(X) \wedge Z\varepsilon\mu_w(Y) \wedge \overrightarrow{\top}(t,w) \geq r)$

It turns out, as first proved in a different context in [64], that $\mu_{\top,r}$ satisfies axioms (RM1-RM5*); we include the proof in our case.

Proposition 81. *Functors $\mu_{\top,r}$ satisfy (RM1)-(RM5*)*

Proof. For (RM1): assume that $X\varepsilon\mu_{\top,1}(Y)$ so for any Z from $Z\varepsilon\mu_u(X), Z\varepsilon\mu_v(Y)$ it follows that $\overrightarrow{\top}(u,v) \geq 1$ hence $\top(1,u) \leq v$ i.e. $u \leq v$; this implies that $\forall Z.(Z\varepsilon\mu_1(X) \Longrightarrow Z\varepsilon\mu_1(Y))$ i.e. $\forall Z.(Z\varepsilon el(X) \Longrightarrow Z\varepsilon el(Y))$ and thus $X\varepsilon el(Y)$. Conversely, $X\varepsilon el(Y)$ implies $\forall Z.(Z\varepsilon\mu_u(X) \wedge Z\varepsilon\mu_v(Y) \Longrightarrow u \leq v$ so $\overrightarrow{\top}(u,v) \geq 1)$ and finally $X\varepsilon\mu_{\top,1}(Y)$.

Concerning (RM2), let $X\varepsilon\mu_{\top,1}(Y)$ and $Z\varepsilon\mu_{\top,u}(X)$ hence for any T from $T\varepsilon\mu_\alpha(Z)$, $T\varepsilon\mu_v(X)$, $T\varepsilon\mu_w(Y)$ it follows that $v \leq w$ hence $\overrightarrow{\top}(\alpha,w) \geq \overrightarrow{\top}(\alpha,v)$ a fortiori $\overrightarrow{\top}(\alpha,v) \geq u$ implies $\overrightarrow{\top}(\alpha,w) \geq u$ i.e. $Z\varepsilon\mu_{\top,u}(Y)$.

In case of (RM3), assume that $X = Y$; then for any T, α : $T\varepsilon\mu_\alpha(X) \iff T\varepsilon\mu_\alpha(Y)$ hence for any T from $T\varepsilon\mu_\gamma(X), T\varepsilon\mu_\delta(Z)$ with $\overrightarrow{\top}(\gamma,\delta) \geq r$ it follows that $T\varepsilon\mu_\gamma(Y), T\varepsilon\mu_\delta(Z)$ with $\overrightarrow{\top}(\gamma,\delta) \geq r$ i.e. $X\varepsilon\mu_{\top,r}(T) \Longrightarrow Y\varepsilon\mu_{\top,r}(T)$.

(RM4) is obviously satisfied by virtue of definition of $\mu_{\top,r}$.

Now, for (RM5): assume that $ext(X,Y)$ hence $X\varepsilon\mu_0^+(Y)$. Let $X\varepsilon\mu_{\top,r}(Y)$; then from $T\ \varepsilon\mu_\gamma(X)$, $T\varepsilon\mu_\delta(Y)$ it follows that $\overrightarrow{\top}(\gamma,\delta) \geq r$ for any T and some γ,δ hence $\top(r,\gamma) \leq \delta$. In particular, for $T = X$, we have $\top(r,1) \leq 0$ i.e. $r \leq 0$.

Consider finally (RM5*): assume $ext(X,Y)$; then $\exists s < 1.X\varepsilon\mu_s^+(Y)$. We have then $X\varepsilon\mu_1(X)$, $X\varepsilon\mu_s^+(Y)$ and $\overrightarrow{\top}(1,s) \geq r$ implies $\top(r,1) \leq s$ i.e. $r \leq s$ so $X\varepsilon\mu_{\top,r}(Y)$ implies $r \leq s$ i.e. $X\varepsilon\mu_{\top,t}^+(Y)$ with $t \leq s$.

An advantage of this rough inclusion is the fact that it does satisfy a deduction rule of the form(DR)$\frac{X\varepsilon\mu_r(Y), Y\varepsilon\mu_s(Z)}{X\varepsilon\mu_u(Z)}$where $u = f(r,s)$ depends functionally

on r,s. Clearly, the obvious candidate for f is $\top$. Again, we include a short proof for completeness sake.

Proposition 82. *(DR) holds in the form :* $\frac{X\varepsilon\mu_{\top,r}(Y), Y\varepsilon\mu_{\top,s}(Z)}{X\varepsilon\mu_{\top,\top(r,s)}(Z)}$.

Proof. Assume that $X\varepsilon\mu_{\top,r}(Y), Y\varepsilon\mu_{\top,s}(Z)$; we have then: $T\varepsilon\mu_\alpha(X)$, $T\varepsilon\mu_\beta(Y)$, $T\varepsilon\mu_\delta(Z)$ imply $\overrightarrow{\top}(\alpha,\beta) \geq r$, $\overrightarrow{\top}(\beta,\delta) \geq s$ i.e. $\top(r,\alpha) \leq \beta, \top(s,\beta) \leq \delta$ and monotonicity of $\top$ implies $\top(s,\top(r,\alpha)) \leq \delta$ so $\top(\top(s,r),\alpha)) \leq \delta$ and $\top(\top(r,s),\alpha)) \leq \delta$ which finally yields $\overrightarrow{\top}(\alpha,\delta) \geq \top(r,s)$ a fortiori $X\varepsilon\mu_{\top,\top(r,s)}(Z)$.

We will exploit this advantage of $\mu_{\top,r}$ in the sequel.

Remark. We may observe that we do not decide the status of
(CRM5) $X\varepsilon\mu_0^+(Y) \Longrightarrow ext(X,Y)$ i.e. the converse to (RM5).

Remark. We may also notice that accepting (RM5) bears inconveniently on $\mu_{\top,r}$ as witnessed by the following

Proposition 83. *Under (RM5), for any* X, Y :

1. $X \varepsilon \mu_{\top,1}(Y)$ *whenever* $X \varepsilon el(Y)$;
2. $X \varepsilon \mu^{+}_{\top,0}(Y)$ *whenever* $non(X \varepsilon el(Y))$.

Proof. Case (1) has been settled already by proving (RM1) for $\mu_\top$; in case (2), there exists Z with $Z \varepsilon el(X), ext(Z, Y)$ i.e. $Z \varepsilon \mu_1(X), Z \varepsilon \mu_0^+(Y)$. As $\vec{\top}(1, 0) \geq r$ implies $\top(r, 1) = r \leq 0$, it follows that $X \varepsilon \mu^{+}_{\top,0}(Y)$.

The moral of the last proposition is that under (RM5), $\mu_\top$ becomes a 0-1 measure discerning only between being an element and not being an element. For this reason, we will apply (RM5*) in the sequel as a more realistic approach.

5.4 Symmetrization

Analogies with distance, imposing themselves for rough inclusions, may be carried further to the point of noticing that our functors μ_r are not (and usually cannot be) symmetric. It might be convenient in some contexts to consider a symmetric counterpart of μ_r, say μ_r^s.

Definition 84. $X \varepsilon \mu_r^s(Y) \iff X \varepsilon \mu_r(Y) \wedge Y \varepsilon \mu_r(X)$.

From our axioms about rough inclusions and earlier results the following properties of functors μ_r^s follow

Proposition 85. *1.* $X \varepsilon \mu_1^s(Y) \iff X \varepsilon = (Y)$;
2. $X \varepsilon \mu_1^s(Y) \Longrightarrow (Z \varepsilon \mu_r^s(Y) \iff Z \varepsilon \mu_r^s(X))$;
3. $X \varepsilon \mu_r^s(Y) \iff Y \varepsilon \mu_r^s(X)$;
4. $X \varepsilon \mu_r^s(Y) \wedge t \leq r \Longrightarrow X \varepsilon \mu_t^s(Y)$;
5. *(RM5)* $\Longrightarrow (ext(X, Y) \Longrightarrow X \varepsilon \mu_0^{s+}(Y))$;
6. *(RM5*)* $\Longrightarrow (ext(X, Y) \Longrightarrow \exists t < 1. X \varepsilon \mu_t^{s+}(Y))$;
7. *(DR)* $\frac{X \varepsilon \mu^s_{\top,r}(Y), Y \varepsilon \mu^s_{\top,s}(Z)}{X \varepsilon \mu^s_{\top,\top(r,s)}(Z)}$.

We now propose to synthesize basic constructs applied in Qualitative Spatial Reasoning based currently on Connection approach in Mereology by means of Rough Mereology, a paradigm for approximate reasoning based on functors of being a part in a degree. Rough Mereology is conceived as a hybridization of ideas of Rough Set Theory and Mereology and it proposes a model for approximate reasoning in which basic constructs are derivable from data.

6 Introduction to Qualitative Spatial Reasoning

Qualitative Reasoning aims at studying concepts and calculi on them that arise often at early stages of problem analysis when one is refraining from qualitative or metric details cf. [16]; as such it has close relations to the design cf. [11] as well as planning stages cf. [31] of the model synthesis process. Classical formal approaches to spatial reasoning i.e. to representing spatial entities (points, surfaces, solids etc.) and their features (dimensionality, shape, connectedness degree etc.) rely on Geometry or Topology i.e. on formal theories whose models are spaces (universes) constructed as sets of points; contrary to this approach, qualitative reasoning about space often exploits pieces of space (regions, boundaries, walls, membranes etc.) and argues in terms of relations abstracted from a commonsense perception (like *connected*, *discrete from*, *adjacent*, *intersecting*). In this approach, points appear as ideal objects (e.g. ultrafilters of regions/solids [88]). Qualitative Spatial Reasoning has a wide variety of applications, among them, to mention only a few, representation of knowledge, cognitive maps and navigation tasks in robotics (e.g. [42], [43], [44], [1], [3], [23], [39], [28]), Geographical Information Systems and spatial databases including *Naive Geography* (e.g. [26], [27], [35], [24]), high-level Computer Vision (e.g. [94]), studies in semantics of orientational lexemes and in semantics of movement (e.g. [6], [5]). Spatial Reasoning establishes a link between Computer Science and Cognitive Sciences (e.g. [29]) and it has close and deep relationships with philosophical and logical theories of space and time (e.g. [70], [9], [2]). A more complete perspective on Spatial Reasoning and its variety of themes and techniques may be acquired by visiting one of the following sites : [85], [93], [58].

Any formal approach to Spatial Reasoning, however, would require a formal approach to Ontology as well cf. [34], [80], [12]. In this paper we adopt as formal Ontology the ontological theory of Stanisław Leśniewski (cf. [51], [52], [79], [49], [38], [20]. This theory is briefly introduced in Section 3.

For expressing relations among entities, mathematics proposes two basic languages: the language of set theory, based on the opposition element–set, where distributive classes of entities are considered as sets consisting of discrete atomic entities, and languages of mereology, for discussing entities continuous in their nature, based on the opposition part–whole. It is thus not surprising that Spatial Reasoning relies to great extent on mereological theories of part cf. [4], [5], [6], [14], [17], [32], [33], [30], [81], [82], [57].

Mereological ideas have been early applied toward axiomatization of geometry of solids cf. [47], [88]. Mereological theories dominant nowadays come from ideas proposed independently by Stanisław Leśniewski and Alfred North Whitehead.

Mereological theory of Leśniewski is based on the notion of a part (proper) and the notion of a (collective) class cf. [51], [53], [20], [83], [54].

Mereological ideas of Whitehead based on the dual to part notion of an extension [95] were formulated as the Calculus of Individuals [48] and given a formulation in terms of the notion of a Connection [14]. Mereology based on connection gave rise to spatial calculi based on topological notions derived therefrom (mereotopology) cf. [18], [16], [22], [25], [5], [6], [17], [32], [33], [30], [81], [57].

In this Chapter, we are adopting mereological theory of Stanisław Leśniewski formalized according to His program within His Ontology. See Section 4 for the details.

We study in this Chapter possible applications of Rough Mereology to Spatial Reasoning in the frame of Information Systems. We demonstrate that in the framework of Rough Mereology one may define a quasi–Čech topology [21] (a quasi– topology was introduced in the connection model of Mereology [14], [5] under additional assumptions of regularity); see Section 7 for this topic.

We apply mereotopology defined via rough mereological notions to a study of connections. We introduce in Section 8 some notions of a connection induced from rough inclusions and we demonstrate that they induce the original mereological notion of an element. This shows that Rough Mereology offers a reasoning mechanism about spatial relations which contains calculi based on connection.

Finally, we apply Rough Mereology toward inducing geometrical notions. It is well known [89], [9] that geometry may be introduced via notions of nearness, betweenness etc. In Section 9, we define these notions by means of a rough mereological notion of distance and we show that in this way a geometry may be defined in the rough mereological universe.

As rough mereological constructs (rough inclusions) may be induced from data tables (information systems) as indicated in Section 4 reasoning about spatial entities by means of Rough Mereology may be carried out on the basis of data (e.g. spatial databases).

6.1 Mereology via Connection

This approach [95], [48], [14] is based on the functor of being connected; for the uniformity of exposition sake, we will formulate all essentials of this theory in the ontological language applied above.

The requirements for a functor C of connection are as follows.

Definition 86. (C1) $X \varepsilon C(Y) \Longrightarrow X \varepsilon X \wedge Y \varepsilon Y$.
Asserting that C is defined on individuals.
(C2) $X \varepsilon C(X)$.
Asserting reflexivity of C.
(C3) $X \varepsilon C(Y) \Longleftrightarrow Y \varepsilon C(X)$.
Asserting that C is symmetric.
(C4) $\forall Z (Z \varepsilon C(X) \Longleftrightarrow Z \varepsilon C(Y)) \Longrightarrow X \varepsilon = (Y)$.
Asserting extensionality of C.

It follows that any connection functor C is a reflexive and symmetric functor on individuals with the extensionality property: entities connected with the same entities are identical (notice that some schemes for connection calculi dispense with extensionality cf. [18], [33], [57]).

From the functor C, other functors are derived; we recall the most important of them now.

Definition 87. The following lists basic functors dependent on C.

1. $X\varepsilon DC(Y) \iff non(X\varepsilon C(Y))$ (*X is disconnected from Y*);
2. $X\varepsilon P(Y) \iff \forall Z(Z\varepsilon C(X) \Longrightarrow Z\varepsilon C(Y))$ (*X is an element of Y*);
3. $X\varepsilon PP(Y) \iff X\varepsilon P(Y) \wedge non(Y\varepsilon P(X))$ (*X is a part of Y*);
4. $X\varepsilon O(Y) \iff \exists Z(Z\varepsilon P(X) \wedge Z\varepsilon P(Y))$ (*X, Y overlap*);
5. $X\varepsilon EC(Y) \iff X\varepsilon C(Y) \wedge non(X\varepsilon O(Y))$ (*X is externally connected to Y*);
6. $X\varepsilon TPP(Y) \iff X\varepsilon PP(Y) \wedge \exists Z(Z\varepsilon EC(X) \wedge Z\varepsilon EC(Y))$ (*X is a tangential part of Y*);
7. $X\varepsilon NTPP(Y) \iff X\varepsilon PP(Y) \wedge non(X\varepsilon TPP(Y))$ (*X is a non-tangential part of Y*).

Connection allows for a variety of functors of topological characters (one may define a quasi-topological interior by means of $NTPP$ cf. eg. [5], [6], [14], [57]).

We return to this topic in the next Section devoted to mereotopology.

7 Mereotopology

We now are concerned with topological structures arising in mereological universe endowed with a rough inclusion.

As mentioned few lines above, topological structures may be defined within the connection framework via the notion of a non-tangential part. Interior entities are formed then by means of some fusion operators cf. e.g. [5], [57]. The functor of connection allows also for some calculi of topological character based directly on regions e.g. $RCC - calculus$ cf. [33]. For a different approach where connection may be derived from the axiomatized notion of a boundary cf. [82].

These topological structures provide a mereotopological environment in which it is possible to carry out spatial reasoning (cf. op.cit., op.cit.). We now demonstrate that in rough merelogical framework one defines in a natural way Čech topologies i.e. topologies which may be deficient in that the closure operator may not have some properties of the topological closure (viz., idempotency and the finite intersection property).

7.1 Mereotopology : Čech topologies

It has been demonstrated that in mereological setting a quasi–Čech topology may be defined (cf. [14]) which under additional artificial assumptions (op.cit.) may be made into a quasi–topology. Here, we induce a quasi–Čech topology (i.e. topology without the null object) in any rough mereological universe.

We would like to recall that a topology on a given domain U may be introduced by means of a closure operator cl satisfying the *Kuratowski axioms* [46]:

1. (Cl1) $cl\emptyset = \emptyset$;
2. (Cl2) $clclX = clX$;
3. (Cl3) $X \subseteq clX$;

4. (Cl4) $cl(X \cup Y) = clX \cup clY$.

The dual operator *int* of *interior* is then defined by means of the formula: $intX = U - cl(U - X)$ and it has properties expressed by this duality: $int\emptyset = \emptyset$, $intintX = intX$, $intX \subseteq X$, $int(X \cap Y) = intX \cap intY$.

The Čech topology [21] is a weaker structure as it is required here only that the closure operator satisfy the following:

1. (ČCl1) $cl\emptyset = \emptyset$;
2. (ČCl2) $X \subseteq clX$;
3. (ČCl3) $X \subseteq Y \Longrightarrow clX \subseteq clY$.

so the associated Čech interior operator *int* should only satisfy the following: $int\emptyset = \emptyset$; $intX \subseteq X$; $X \subseteq Y \Longrightarrow intX \subseteq intY$.

Čech topologies arise naturally in problems related to information systems when one considers coverings induced by similarity relations instead of partitions induced by indiscernibility relations [55].

We now introduce a quasi–Čech topology into a rough mereological universe: we may remember that in our context of Ontology, the empty set (name) may not be used.

To this end, we define the class $Kl_r X$ for any object X and $r < 1$ by the following. First, we introduce a name $(M_r X)$ for the property of being a part in a degree r.

Definition 88. $Z\varepsilon M_r X \Longleftrightarrow Z\varepsilon Z \wedge Z\varepsilon\mu_r(X)$

Now, we define the individual entity $Kl_r X$.

Definition 89. $Z\varepsilon Kl_r X \Longleftrightarrow Z\varepsilon Z \wedge Z\varepsilon M_r X$.

Thus $Kl_r X$ is the class of objects having the property $\mu_r(X)$. We now give a direct characterization of $Kl_r X$. With this aim, we introduce a name $B_r X$ defined by means of the condition:

$$Z\varepsilon el(B_r X) \Longleftrightarrow \exists T(Z\varepsilon el(T) \wedge T\varepsilon\mu_r(X)).$$

This definition is correct, as $B_r X = Kl(el(B_r X))$. Then we have

Proposition 90. $Kl_r X = B_r X$

Proof. Assume first that $Z\varepsilon el(Kl_r X)$; then by the class definition (Definition 44), for some U, W, we have $U\varepsilon el(Z)$, $U\varepsilon el(W)$, $W\varepsilon\mu_r(X)$, hence $U\varepsilon el(B_r X)$ and the inference rule (Proposition 61) implies that $Kl_r X\varepsilon el(B_r X)$.

Conversely, $Z\varepsilon el(B_r X)$ implies that for some T we have $Z\varepsilon el(T)$, $T\varepsilon\mu_r(X)$ hence by he class definition $T\varepsilon el(Kl_r X)$ and so $Z\varepsilon el(Kl_r X)$ implying by the inference rule that $B_r X \ \varepsilon \ el(Kl_r X)$. Hence $Kl_r X = B_r X$.

From this the following corollary follows

Corollary 91. *For* $s \leq r$, $Kl_r X \varepsilon el(Kl_s X)$.

Indeed, $Z\varepsilon el(Kl_r X)$ means that $Z\varepsilon el(T), T\varepsilon\mu_r(X)$ for some T so by (RM4) we have $T\varepsilon\mu_s(X)$ and thus $Z\varepsilon el(Kl_s X)$. The corollary follows.

We mention yet a monotonicity property.

Proposition 92. $X\varepsilon el(Y) \Longrightarrow Kl_r X \varepsilon el(Kl_r Y)$.

Example 2. We recall the Table $Binary1$ from Example 1. With the notation of that Example, we find the class $Kl_{0.5}(X)$ for $X = \{u_1, u_2, u_6\}$; we mark this class in the Table $Binary2$ below with boldface.

	a_1	a_2	a_3
u_1	**1**	**0**	**1**
u_2	**1**	**0**	**0**
u_3	**1**	**1**	**0**
u_4	0	1	1
u_5	0	1	0
u_6	**1**	**0**	**1**
u_7	**1**	**1**	**0**

Table 2. *Binary2*: The class $Kl_{0.5}(X)$

We admit B defined as follows as a base for open sets with which we define the interior operator.

Definition 93. $Z\varepsilon B \Longleftrightarrow Z\varepsilon Z \wedge \exists X, r < 1.Z\varepsilon Kl_r X$.

Following this we define a new functor *int*. Again, we introduce first a new name $I(X)$.

Definition 94. $Z\varepsilon I(X) \Longleftrightarrow Z\varepsilon Z \wedge \exists s < 1(Kl_s Z\varepsilon el(X))$.

Now, we define *int*.

Definition 95. $int(X) = Kl(I(X))$.

Then we have the following properties of *int*.

Proposition 96. *For any* X, Y :

1. $int(X) \varepsilon el(X)$;
2. $X \varepsilon el(Y) \Longrightarrow int(X) \varepsilon el(int(Y))$;

Proof. For (1): assume that $Z \varepsilon el(int(X))$; there exist U, W with $U \varepsilon el(Z)$, $U \varepsilon el(W)$, $Kl_s W \varepsilon el(X)$ for some $s < 1$; hence, $W \varepsilon el(X)$ and $U \varepsilon el(X)$ so the inference rule (Proposition 61) implies that $int(X) \varepsilon el(X)$.

In case (2), assume that $X \varepsilon el(Y)$ and let $Z \varepsilon el(int(X))$. We have U, W with $U \varepsilon el(Z)$, $U \varepsilon el(W)$, $Kl_s W \varepsilon el(X)$ for an $s < 1$ hence $Kl_s W \varepsilon el(Y)$ and thus $W \varepsilon I(Y)$ hence $W \varepsilon el(int(Y))$ so a fortiori $U \varepsilon el(int(Y))$ so the inference rule implies that $int(X) \varepsilon el(int(Y))$.

Properties (1)-(2) witness that the quasi–topology introduced by B is a *quasi–Čech topology.* We denote it by the symbol τ_μ.

Proposition 97. *Rough mereotopology τ_μ induced by the rough inclusion μ_r is a quasi–Čech topology.*

Remark. Let us observe that under (RM5) the topology τ_μ is discrete: every individual is an open set (a singleton).

We now study the case of mereotopology under functors $\mu_{\top,r}$; in this case, the quasi– Čech topology τ_μ turns out to be a quasi–topology.

7.2 Mereotopology: the case of $\mu_\top$

We begin with an application of deduction rule (DR). We denote by the symbol $Kl_{\top,r}X$ the set $Kl_r X$ in case of the rough inclusion $\mu_\top$. We assume that $\top(r,s) < 1$ when $rs < 1$. We propose a new direct characterization of $Kl_{\top,r}X$.

Proposition 98. $Z \varepsilon el(Kl_{\top,r}X) \Longleftrightarrow Z \varepsilon \mu_{\top,r}(X)$.

Proof. By Proposition 90, $Z \varepsilon el(Kl_{\top,r}X)$ means that $Z \varepsilon el(T)$, $T \varepsilon \mu_{\top,r}(X)$ for some T hence $Z \varepsilon \mu_{\top,1}(T)$, $T \varepsilon \mu_{\top,r}(X)$ imply by (DR) that $Z \varepsilon \mu_{\top,\top(1,r)}(X)$ i.e. $Z \varepsilon \mu_{\top,r}(X)$.

This Proposition means that $Kl_{\top,r}X$ may be regarded as "*an open ball of radius r centered at X*".

We assume now, additionally, that the t-norm $\top$ has the property that: for every $r < 1$ there exists $s < 1$ such that $\top(r,s) \geq r$. With this assumption, we have the following.

Proposition 99. *For $Z \varepsilon el(Kl_{\top,r}(X))$,*
if $s_0 = \arg\_\min\{s : \top(r,s) \geq r\}$ *then* $Kl_{\top,s_0}(Z) \varepsilon el(Kl_{\top,r}(X))$.

Proof. Let $s \geq s_0$; consider $T \varepsilon el(Kl_{\top,s}(Z))$ so $T \varepsilon \mu_{\top,s}(Z)$. Then $T \varepsilon \mu_{\top,\top(s,r)}(X)$ hence $T \varepsilon \mu_{\top,r}(X)$ so $T \varepsilon el(Kl_{\top,r}(X))$ implying finally by the inference rule (Proposition 61) that $Kl_{\top,s_0}(Z) \varepsilon el(Kl_{\top,r}(X))$.

We define a functor of two nominal individual variables Ov (of rough mereological overlap) and a functor of two nominal individual variables AND.

Definition 100. 1. $Ov(X,Y) \Longleftrightarrow \exists Z. Z\varepsilon el(X) \wedge Z\varepsilon el(Y)$;
2. $Z\varepsilon el(AND(X,Y)) \Longleftrightarrow Ov(X,Y) \wedge Z\varepsilon el(X) \wedge Z\varepsilon el(Y)$.

Proposition 101. *The rough mereotopology $\tau_{\mu_\top}$ has the property:* $AND(int(X), int(Y)) = int(AND(X,Y))$ *holds whenever* $AND(int(X), int(Y))$ *is non-empty.*

Proof. The intersection of two open basic classes may be described effectively by means of Proposition 99 : assume that $Z\varepsilon el(Kl_{\top,r}(X))$ and $Z\varepsilon el(Kl_{\top,s}(Y))$ for some $r,s<1$. Then for $t_0 = \arg\_\min\{t : \top(r,t) \geq r, \top(s,t) \geq s\}$ and $1 > t \geq t_0$, we have $Kl_{\top,t}(Z)\varepsilon el(Kl_{\top,r}(X))$ and $Kl_{\top,t}(Z)\varepsilon el(Kl_{\top,s}(Y))$. The general case follows easily.

Finally, we check that under our assumptions, the operator of interior is idempotent: $intint = int$ which will conclude our verification that the rough mereological topology is a topology.

Proposition 102. $int(int(X)) = int(X)$.

Proof. It suffices to show that $int(X)\varepsilon el(int(int(X)))$ so we consider Z with $Z\varepsilon el(int(X))$. For some U,W, we have $U\varepsilon el(Z)$, $U\varepsilon el(W)$, $Kl_sW\varepsilon el(X)$, some $s<1$.

Now, we check that: $Kl_sW = int(Kl_sW)$; we consider then P with $P\ \varepsilon\ el(Kl_sW)$. There exist R,Q with $R\varepsilon el(P)$, $R\varepsilon el(Q)$, $Q\varepsilon\mu_sW$. It follows that $R\varepsilon\mu_sW$ hence $R\varepsilon el(kl_sW)$ so $Kl_tR\varepsilon Kl_sW$ for some $t<1$ and finally $R\ \varepsilon\ el(int(Kl_sW))$. By the inference rule (Proposition 61), $Kl_sW\varepsilon el(int(Kl_sW))$ and the identity follows.

Returning to the main proof, as $Kl_sW\varepsilon el(X)$, we have $int(Kl_sW)\varepsilon el(int(X))$ hence $Kl_sW\ \varepsilon el(int(X))$ and thus $Kl_tU\varepsilon el(int(X))$ for some $t<1$ so, finally, $U\varepsilon el(int(int(X)))$ and the inference rule shows that $int(X)\varepsilon el(int(int(X)))$.

Corollary 103. *The rough mereological topology induced by the rough inclusion* $\mu_{\top,r}$ *is a quasi–topology.*

8 Connections from Rough Inclusions

In Section 6.1, we presented basic notions related to mereological theories based on the notion of a connection. We recall that a connection is a functor which satisfies axioms (C1)-(C4) of Section .

In this section we will investigate some methods for inducing connections from rough inclusions. Clearly, the presence of topology induced in the preceding section allows for a few approaches to this problem. We begin with a notion of a connection in a strong sense.

8.1 Strong connection

We define a name–forming functor C_T on individual entities as follows.

Definition 104. $X\varepsilon C_T(Y) \iff X\varepsilon X \wedge non(\exists r, s < 1.ext(Kl_r X, Kl_s Y))$.

Thus, X and Y are connected in the strong sense whenever they cannot be separated by means of their open neighborhoods

We check whether C_T thus defined does satisfy (C1)-(C4). It may be clear that (C1), (C2), (C3) hold irrespective of properties of μ. The status of (C4) will clearly depend on our assumed functor μ. In case $non(X = Y)$, we have e.g. $Z\varepsilon el(X), ext(Z, Y)$ with some Z. Clearly, $Z\varepsilon C_T(X)$; to prove that $non(Z\varepsilon C_T(Y))$, we need some assumptions about the form of μ.

Proposition 105. *Assume (RM5*) and consider $\mu^s_\top$ with a t-norm $\top$ which would satisfy the following: given $s < 1$, there exist $\alpha, \beta < 1$ with the property that $\top(\alpha, \beta) > s$. Then, C_T induced via $\mu^s_\top$ would satisfy (C4).*

Proof. We need only to check that $non(Z\varepsilon C_T(Y))$ whenever $ext(Z, Y)$; by (RM 5*), $Z\varepsilon\mu^{s+}_{\top,s}(Y)$ for some $s < 1$. Assume that $non(ext(Kl_\alpha Y, Kl_\beta Z)$ for some $\alpha, \beta < 1$; there is W with $W\varepsilon el(Kl_\alpha Y)$, $W\varepsilon el(Kl_\beta Z)$ hence $W\varepsilon\mu^s_{\top,\alpha}(Y)$,

$W\varepsilon\mu^s_{\top,\beta}(Z)$ and thus $Z\varepsilon\mu^s_{\top,\beta}(W)$ so $Z\ \varepsilon\ \mu^s_{\top,\top(\alpha,\beta)}(Y)$; taking $\alpha, \beta < 1$ with $\top(\alpha, \beta) > s$, we arrive at a contradiction with $Z\varepsilon\mu^{s+}_{\top,s}(Y)$. Therefore, $non(Z\varepsilon C_T(Y))$.

In connection framework, the notion of an element is derived from the functor C of connection; the resulting functor of an element is denoted here by the symbol el_C. We will find relationships between the original functor el of an element and the functor el_C. To this end, we have

Proposition 106. *For any functor of the form $\mu_\top$: $X\varepsilon el(Y) \Longrightarrow X\varepsilon el_{C_T}(Y)$.*

Proof. From $Z\varepsilon C_T(X)$ it follows that $non(ext(Kl_\alpha X, Kl_\beta Z))$ for any $\alpha, \beta < 1$; as $Kl_{\top,s}X\ \varepsilon\ el(Kl_{\top,r}Y)$ for every $s \geq \arg\_\min\{\top(r, s) \geq r\}$, the claim follows.

Proposition 107. *For any functor of the form $\mu^s_\top$: $X\varepsilon el_{C_T}(Y) \Longrightarrow X\varepsilon el(Y)$.*

Proof. Clearly, $non(X\varepsilon el(Y))$ implies $Z\varepsilon el(X)$, $ext(Z, Y)$ with some Z which imply $C_T(Z, X)$, $non(C_T(Z, Y))$ i.e. $non(X\varepsilon el_C(Y))$.

Corollary 108. *For any functor of the form $\mu^s_\top$: $X\varepsilon el_{C_T}(Y) \iff X\varepsilon el(Y)$.*

We may therefore create in the framework of rough mereology an alternative scheme of calculus of individuals based on the connection C_T inducing the same notion of an element as the original mereological one.

As in our model topological structures arise in a natural way via "metrics" μ, we may afford a more stratified approach to connection and separation properties. So we propose a notion of a graded connection $C(r, s)$.

8.2 From Graded Connections to Connections

We begin with a definition of an individual entity $Bd_r X$.

Definition 109. $Bd_r X \varepsilon Kl(\mu_r^+ X)$ where $Z\varepsilon\mu_r^+(X) \Longleftrightarrow Z\varepsilon\mu_r(X) \wedge non(\exists s > r.Z\varepsilon\mu_s(X))$.

and then we introduce a *graded* (r,s)*-connection* $C(r,s)$ $(r,s<1)$ via

Definition 110. $X\varepsilon C(r,s)(Y) \Longleftrightarrow X\varepsilon X \wedge \exists W.W\varepsilon el(Bd_r X) \wedge W\varepsilon el(Bd_s Y)$.

We have then clearly:

Proposition 111. *1.* $X\varepsilon C(1,1)(X)$;
2. $X\varepsilon C(r,s)(Y) \Longrightarrow Y\varepsilon C(s,r)(X)$.

Concerning the property (C4), we adopt here a new approach. It is valid from theoretical pont of view to assume that we may have "infinitesimal " parts i.e. objects as "small" with respect to μ as desired cf. a similar assumption in Mereology based on Connection about "infinite divisibility" [57], [18].

Infinitesimal parts model We adopt a new axiom of infinitesimal parts

(IP) $non(X\varepsilon el(Y)) \Longrightarrow \forall r>0.\exists Z\varepsilon el(X), s<r.Z\varepsilon\mu_s^+(Y)$.

Our rendering of (C4) under (IP) is as follows:

Proposition 112.

$$non(X\varepsilon el(Y)) \Longrightarrow \forall r>0.\exists Z, s<r.Z\varepsilon C(1,1)(X) \wedge Z\varepsilon C(1,s)(Y).$$

Remark. It may be helpful in practical applications to have a "threshold" variant of (IP) i.e. to require for a certain threshold δ that $Z\varepsilon\mu_s^+(Y)$ with $s \leq \delta$.

Connections from Graded Connections Our notion of a connection will depend on a threshold, α, set according to the needs of the context of reasoning. Given $0<\alpha<1$, we define a connection functor C_α as follows.

Definition 113. $X\varepsilon C_\alpha(Y) \Longleftrightarrow X\varepsilon X \wedge \exists r,s \geq \alpha.X\varepsilon C(r,s)(Y)$.

Then we have

Proposition 114. *Assume (IP). For any* α *:*

1. $X\varepsilon C_\alpha(X)$;
2. $X\varepsilon C_\alpha(Y) \Longrightarrow Y\varepsilon C_\alpha(X)$;
3. $X \neq Y \Longrightarrow$
   $\exists Z.(Z\varepsilon C_\alpha(X) \wedge non(Z\varepsilon C_\alpha(Y)) \vee Z\varepsilon C_\alpha(Y) \wedge non(Z\varepsilon C_\alpha(X)))$.

Thus the functor C_α has all the properties of a connection.

Restoring rough mereology from connections We show now that when we adopt mereological notions as they are defined via connections in mereological calculi, we do not get anything new: we come back to rough mereology we started from i.e. the case here is analogous to the case of C_T under more specialized functors $\mu^s_\top$. We claim

Proposition 115. *Assume (IP). Then*

$$X\varepsilon el_{C_\alpha}(Y) \Longleftrightarrow X\varepsilon el(Y).$$

Proof. Clearly, $X\varepsilon el_{C\alpha}(Y) \Longrightarrow X\varepsilon el(Y)$. Assume that $X\varepsilon el(Y)$; $Z\varepsilon C_\alpha(X)$. There is W with $W\varepsilon\mu^+_r(Z)$ and $W\varepsilon\mu^+_s(X), r, s \geq \alpha$. Then by (RM2), $W\varepsilon\mu^+_{s'}(Y)$ with an $s' \geq s$ and so $Z\varepsilon C_\alpha(Y)$. It follows that $X\varepsilon el(Y) \Longrightarrow X\varepsilon el_{C_\alpha}(Y)$.

Any of connections C_α restores thus the original notion of an element, el.

We have in consequence two schemes for introducing connections compatible with rough mereology defined with respect to a given mereology: a connection C_T defined under a specialization of functors μ_r to the symmetric renormalized form $\mu^s_\top$ and connections C_α defined for any functor μ_r under an additional axiom (IP).

Connection C_T may be regarded as a limit case of connections C_α and it seems an interesting problem to describe the class of models for C_T where it does not coincide with overlap (see, however, Section 10.1).

Rough Mereological Interior vs.Connection Interior : a Comparison We will make use of the functor $NTPP$ (non-tangential part) cf. Section 6.1 in order to recall the definition of the connection-induced interior operator int_C cf. [57], [5], [6], [14]. Our definition below is correct because of the following monotonicity property

Proposition 116. *For C either C_T or C_α, we have $X\varepsilon C(Y) \wedge Y\varepsilon el(Z) \Longrightarrow X\varepsilon C(Z)$*

Definition 117. $Z\varepsilon el(int_C(X)) \Longleftrightarrow Z\varepsilon NTPP(X)$.

We now demonstrate that - under appropriate circumstances - the connection–defined interior int_C is related closely to the topological interior int defined above in the rough mereological setting. In the proofs that follow, we refer to functors O of overlap and EC of external connection defined in Section 6.1.

Proposition 118. *For any functor of the form $\mu^s_\top$: $int(X)\varepsilon(el(int_{C_T}X))$.*

Proof. Assume that $Y \varepsilon el(int(X))$ so there exists $Z \varepsilon el(Y)$, such that for some W, we have $Z \varepsilon el(W)$, $Kl^{s}_{\top,r} W \varepsilon el(X)$ for some $r < 1$ hence $Kl^{s}_{\top,s} Z \varepsilon el(X)$ for some $s < 1$. Had we $non(Z \varepsilon el(int_{C_T} X))$ then there would be T such that: $ext(T, X), Kl^{s}_{\top,\alpha} T \varepsilon O(Kl_{\beta} Z)$ for any pair $\alpha, \beta < 1$. Then, $T \varepsilon \mu^{s}_{\top,\top(\alpha,\beta)}(Z)$ and for $\top(\alpha, \beta) \geq s$, we would have a contradiction. Thus, $Z \varepsilon el(int_{C_T} X)$ and finally, $int(X) \varepsilon el(int_{C_T} X)$.

Proposition 119. *For any* $\mu : (int_{C_\alpha} X) \varepsilon el(int(X))$.

Proof. Assume that $Y \varepsilon el(int_{C_\alpha} X)$; then there exists $Z \varepsilon el(Y)$ such that for some W, we have $Z \varepsilon el(W)$ and $W \varepsilon NTPP(X)$ i.e. $W \varepsilon el(X)$ and for any T :

$non(T \varepsilon EC_\alpha(X) \wedge\ T \varepsilon EC_\alpha(W))$. Assume that $non(Z \varepsilon el(int(X)))$ hence for any $r < 1$, there exists Q_r with $ext(Q_r, X)$, $Q_r \varepsilon el(Kl_r Z)$ hence $Q_r \varepsilon el(Kl_r X)$. Thus, $Q_r \varepsilon C(1, r)(Z)$, $Q_r \varepsilon C(1, r)(X)$;as $Z \varepsilon el(W)$, it follows $Q_r \varepsilon C(1, r)(W)$. Those relations imply that for $r > \alpha$, we have $Q_r \varepsilon EC_\alpha(X)$ and $Q_r \varepsilon EC_\alpha(W)$, contrary to the choice of W. It finally follows that $Z \varepsilon el(int(X))$ and by the inference rule $int_{C_\alpha} X \varepsilon el(int(X))$.

Corollary 120. *For any* μ *of the form* $\mu^{s}_{\top}$:
$(int_{C_\alpha}(X)) \varepsilon el(int(X)) \wedge int(X) \varepsilon el(int_{C_T} X)$.

Thus, the rough mereotopological interior $int(X)$ is delineated by $int_{C_\alpha}(X)$ from below (for any μ) and by $Int_{C_T}(X)$ from above (in case of $\mu^{s}_{\top}$).

9 Mereogeometry

Functors μ_r may be regarded as weak metrics also in the context of geometry. From this point of view, we may apply μ in order to define basic notions of geometry. It is well-known (cf. [90], [9]) that the geometry of euclidean spaces may be based on some postulates about the basic notions of a point and the ternary equidistance functor. In [90] postulates for euclidean geometry over a real-closed field were given based on the functor of betweenness and the quaternary equidistance functor. Similarly, in [9], a set of postulates aimed at rendering general geometric features of geometry of finite-dimensional spaces over reals has been discussed, the primitive notion there being that of nearness.

Geometrical notions have been applied in e. g. studies of semantics of spatial prepositions [6] and in inferences via cardinal directions cf. e.g [45].

It may not be expected that a geometry induced from a rough mereological context proves to be a euclidean one, however, we demonstrate that we may introduce in the rough mereological context functors of nearness, betweenness and equidistance that satisfy basic postulates about these functors valid in euclidean spaces.

9.1 Rough mereological distance, betweenness

We first introduce a notion of distance in our rough mereological universe by letting

Definition 121. $\kappa_r(X,Y) \Longleftrightarrow r = \min\{u, w : X\varepsilon\mu_u^+(Y) \wedge Y\varepsilon\mu_w^+(X)\}$.

We now introduce the notion of betweenness as a functor $T(X,Y)$ of two individual names; the statement $Z\varepsilon T(X,Y)$ reads as "*Z is between X and Y*".

Definition 122. $Z\varepsilon T(X,Y) \Longleftrightarrow Z\varepsilon Z \wedge \forall W.\kappa_r(Z,W) \wedge \kappa_s(X,W) \wedge \kappa_t(Y,W) \Longrightarrow s \leq r \leq t \vee t \leq r \leq s$.

We check that T satisfies the axioms of Tarski [90] for *betweenness.*

Proposition 123. *The following properties hold:*

1. $Z\varepsilon T(X,X) \Longrightarrow Z = X$ *(identity);*
2. $Y\varepsilon T(X,U) \wedge Z\varepsilon T(Y,U) \Longrightarrow Y\varepsilon T(X,Z)$ *(transitivity);*
3. $Y\varepsilon T(X,Z) \wedge Y\varepsilon T(X,U) \wedge X \neq Y \Longrightarrow Z\varepsilon T(X,U) \vee U\varepsilon T(X,Z)$ *(connectivity).*

Proof. By means of κ, the properties of betweenness in our context are translated into properties of betweenness in the real line which hold by the Tarski theorem [90], Theorem 1.

9.2 Nearness

We may also apply κ to define in our context the functor N of nearness proposed in van Benthem [9].

Definition 124. $Z\varepsilon N(X,Y) \Longleftrightarrow Z\varepsilon Z \wedge (\kappa_r(Z,X) \wedge \kappa_s(X,Y) \Longrightarrow s < r)$.

Then the following hold i.e. N does satisfy all axioms for nearness in [9].

Proposition 125.
1. $Z\varepsilon N(X,Y) \wedge Y\varepsilon N(X,W) \Longrightarrow Z\varepsilon N(X,W)$ *(transitivity)*;
2. $Z\varepsilon N(X,Y) \wedge X\varepsilon N(Y,Z) \Longrightarrow X\varepsilon N(Z,Y)$ *(triangle inequality)*;
3. $non(Z\varepsilon N(X,Z))$ *(irreflexivity)*;
4. $Z = X \vee Z\varepsilon N(Z,X)$ *(selfishness)*;
5. $Z\varepsilon N(X,Y) \Longrightarrow Z\varepsilon N(X,W) \vee W\varepsilon N(X,Y)$ *(connectedness).*

Proof. (4) follows by (RM1); (3) is obvious. In proofs of the remaining properties, we introduce a symbol $\mu(X,Y)$ as that value of r for which $\kappa_r(X,Y)$. Then, for (1), assume that $Z\varepsilon N(X,Y), Y\varepsilon N(X,W)$ i.e. $\mu(Z,X) > \mu(X,Y), \mu(X,Y) > \mu(X,W)$ hence $\mu(Z,X) > \mu(X,W)$ i.e. $Z\varepsilon N(X,W)$. In case (2), $Z\varepsilon N(X,Y)$, $X\varepsilon N(Y,Z)$ mean $\mu(Z,X) > \mu(X,Y)$, $\mu(X,Y) > \mu(Y,Z)$ so $\mu(Z,X) > \mu(Y,Z)$ i.e. $X\varepsilon N(Z,Y)$. Concerning (v), $Z\varepsilon N(X,Y)$ implies that $\mu(Z,X) > \mu(X,Y)$ hence either $\mu(Z,X) > \mu(X,W)$ meaning $Z\varepsilon N(X,W)$ or $\mu(X,W) > \mu(X,Y)$ implying $W\varepsilon N(X,Y)$.

We now may introduce the notion of equidistance in the guise of either a functor $Eq(X,Y)$ or a functor $D(X,Y,Z,W)$ defined as follows.

Definition 126. $Z\varepsilon Eq(X,Y)$

$$\Longleftrightarrow Z\varepsilon Z \wedge (non(X\varepsilon N(Z,Y)) \wedge non(Y\varepsilon N(Z,X)))$$

.

It follows that

Proposition 127. $Z\varepsilon Eq(X,Y) \Longleftrightarrow Z\varepsilon Z \wedge (\forall r.\kappa_r(X,Z) \Longleftrightarrow \kappa_r(Y,Z))$.

We may define a functor of equidistance following Tarski [90].

Definition 128. $D(X,Y,Z,W) \Longleftrightarrow (\forall r.\kappa_r(X,Y) \Longleftrightarrow \kappa_r(Z,W))$.

These functors do clearly satisfy the following (cf. [9], [90]).

Proposition 129. *1.* $Z\varepsilon Eq(X,Y) \wedge X\varepsilon Eq(Y,Z) \Longrightarrow Y\varepsilon Eq(Z,X)$ *(triangle equality);*
2. $Z\varepsilon T(X,Y) \wedge W\varepsilon Eq(X,Y) \Longrightarrow D(Z,W,X,W)$ *(circle property);*
3. $D(X,Y,Y,X)$ *(reflexivity);*
4. $D(X,Y,Z,Z) \Longrightarrow X = Y$ *(identity);*
5. $D(X,Y,Z,U) \wedge D(X,Y,V,W) \Longrightarrow D(Z,U,V,W)$ *(transitivity).*

One may also follow van Benthem's proposal for a betweenness functor defined via the nearness functor as follows

Definition 130. $Z\varepsilon T_B(X,Y) \Longleftrightarrow (\forall W.Z\varepsilon W \vee Z\varepsilon N(X,W) \vee Z\varepsilon N(Y,W)$.

One checks in a straightforward way that

Proposition 131. *The functor T_B of betweenness defined according to Definition 130 does satisfy the Tarski axioms.*

9.3 Points

The notion of a point may be introduced in a few ways; e.g. following Tarski [88], one may introduce points as classes of names forming ultrafilters under the ordering induced by the functor of being an element el. Another way, suitable in practical cases, where the universe, or more generally, each ultrafilter F as above is finite i.e. principal (meaning that there exists an object X such that F consists of those Y's for which $X\varepsilon el(Y)$ holds) is to define points as atoms of our universe under the functor of being an element i.e. we define a constant name AT as follows

Definition 132. $X \varepsilon AT \Longleftrightarrow X \varepsilon X \wedge non(\exists Y.Y \varepsilon el(X) \wedge non(X \varepsilon = (Y))$.

We will refer to such points as to *atomic points*. We adopt here this notion of a point.

Clearly, restricting ourselves to atomic points, we preserve all properties of functors of betweenness, nearness and equidistance proved above to be valid in the universe V.

10 Examples

In this section, we will give some examples related to notions and applications thereof presented in the preceding sections.

Our universe will be selected from a quadtree in the Euclidean plane formed by squares $[k + \frac{i}{2^s}, k + \frac{i+1}{2^s}] \times [l + \frac{j}{2^s}, l + \frac{j+1}{2^s}]$ where $k, l \in Z, i, j = 0, 1, ..., 2^s - 1$ and $s = 0, 1, 2, ...$

The choice of atomic points will depend on the level of granularity of knowledge we assume; we may suppose that our objects are localized in space with a positive degree of uncertainty. We will express this uncertainty assuming that our sensoric system perceives each square X as the square X' whose each side length is that of X plus 2α where $\alpha = 2^{-s}$ for some $s > 1$ (we then express uncertainty as uncertainty of location applying "hazing" of objects cf. [92]). By this assumption, we may restrict ourselves to squares with the side length at least 4α (as smaller squares would be localized with uncertainty too high); in consequence, atomic points will be all squares of the above form having the side length equal to 4α. In our example we let for simplicity $4\alpha = 1$. Our atomic points are therefore squares of the form $[k, k+1] \times [l, l+1]$, $k, l \in \mathbf{Z}$.

We will define functors μ_r by letting

$$X \varepsilon \mu_r(Y) \Leftrightarrow \frac{\lambda(X' \cap Y')}{\lambda(X')} \geq r$$

where X', Y' are enlargements of X, Y defined above and λ is the area (Lebesgue) measure in the two-dimensional plane. We may check straightforwardly that

Proposition 133. *Functors μ_r satisfy (RM1)-(RM4)+(RM5)*.*

Let us remark that this measure is a continuous extension of the measure proposed in [63] for the case of discrete information systems (cf. in this respect [62]).

10.1 Connections

Concerning C_α Applying our notion of the connection C_α defined above, we find that two squares having one side in common (like e.g. $X = [0, 1] \times [0, 1]$ and $Y = [0, 1] \times [1, 2]$ are connected in degree $C(1, 0.3(3))$ (i.e. $X \varepsilon C(1, 0.3(3))(Y)$)

while two squares having one vertex in common e.g. like $X = [0,1] \times [0,1]$ and $Y = [1,2] \times [1,2]$ are connected in degree $C(1, 0.1(1))$ (i.e. $X \varepsilon C(1, 0.1(1))(Y)$). Pairs of disjoint squares like $[0,1] \times [0,1]$ and $[0,1] \times [2,3]$ are connected in degree $C(0.3(3), 0.3(3)$ only. Thus, C_α does not discern among connectedness and disconnectedness in topological sense.

It seems useful in this applicational context to introduce a new connection functor C^1_α viz. $X \varepsilon C^1_\alpha(Y)$ whenever $X \varepsilon C(1,r)(Y) \wedge r \geq \alpha \vee X \varepsilon C(s,1)(Y) \wedge s \geq \alpha$.

We may therefore characterize connected regions constructed of atomic squares as such that in any of these regions R, for any two squares X, Y there exists a sequence $X_0 = X, X_1, ..., X_k = Y$ of squares with the property that $X_i \varepsilon C^1_\alpha(X_{i+1})$ with $\alpha > 0$, each $i \leq k-1$.

Shape recognition of connected regions may be carried out also by means of C_α. Consider e.g. a flat torus consisting e.g. of the squares: $[0,1] \times [i, i+1]$ with $i = 0,1,2,3$, $[j, j+1] \times [0,1]$ with $j = 1,2,3,4,5$, $[3,4] \times [j, j+1]$ with $j = 1,2,3,4,5$, $[5,6] \times [1,2]$, $[5,6] \times [2,3]$. Then its cells $X_0, X_1, ..., X_{15}$ satisfy (after the appropriate substitution) the formula:

$$\wedge_{i=1,..,15}(X_{i(mod16)} \varepsilon C^1_{0.3(3)}(X_{i+1(mod16)}) \wedge (X_1 \varepsilon C^1_{0.1(1)}(X_{15})) \wedge (X_4 \varepsilon C^1_{0.1(1)}(X_6))$$

$$\wedge (X_7 \varepsilon C^1_{0.1(1)}(X_9)) \wedge X_{12} \varepsilon C^1_{0.1(1)}(X_{14}))$$

and any collection (name) of squares satisfying this formula will give a region (i.e. the class of this name) having this shape. This idea may be extended to a predicate calculus of (equivalent) shapes.

Concerning C_T Consider a distributed system M_A as proposed in [64], [65], [75], [66], [67] where $Ag = \{ag^0_0\} \cup \bigcup_{i,j=1}^{\infty} \{ag^i_j\}$. *Link* consists of words of the form $ag^{j_1}_{i_1} ag^{j_2}_{i_2} ... ag^{j_k}_{i_k} ag^j_i$ with $i_1 \leq i_2 \leq .... \leq i_k < i$ i.e. any agent ag^j_i of the level i may form a local team with agents of lower levels $i_1, i_2, ..., i_k$. For any agent $ag = ag^j_i$ of the level i, the universe U_{ag} consists of points i.e. squares of side length 2^{-i} and of classes of squares sent by agents in local teams with the head ag. Thus, any agent has a potentially infinite universe. In particular, the agent ag^0_0 has in its universe all points of side length 1 as well as all classes (unions) of finite collections of squares of smaller size sent by lower level agents.

We define the rough inclusion μ_{ag} at any agent ag by the formula:

$X \varepsilon \mu_{ag,r}(Y) \Leftrightarrow \frac{\lambda(X \cap Y)}{\lambda(X)} \geq r$

(i.e. without hazing) and el is $\subseteq$. Then one checks in a direct way that two squares whose union is connected topologically (e.g. $X = [0,1] \times [0,1], Y = [1,2] \times [0,1], X = [0,1] \times [0,1], Y = [1,2] \times [1,2]$) satisfy the formula $X \in C_T(Y)$ (without overlapping i.e. having an element in common) while for X, Y with $Kl(X,Y) = X \cup Y$ not connected topologically we have $non(X \in C_T(Y))$.

10.2 Mereogeometry

We will adopt the notion of betweenness T_B based on the nearness functor. Then we find that e.g. the following triples (X, Z, Y) do satisfy the formula $Z\varepsilon T_B(X, Y)$: $([0,1] \times [0,1], [1,2] \times [0,1], [2,3] \times [0,1])$, $([0,1] \times [0,1], [0,1] \times [1,2], [0,1] \times [2,3])$, $([0,1] \times [0,1], [1,2] \times [1,2], [2,3] \times [2,3])$, $([2,3] \times [2,3], [1,2] \times [1,2], [0,1] \times [0,1])$, $([0,1] \times [0,1], [1,2] \times [0,1], [2,3] \times [1,2])$; clearly, all translates over the digital space $\mathbf{Z^2}$ of the above triples as well as all their rotations by a multiplicity of $\pi/2$ preserve the functor T_B.

The equidistance functor E may be used to define spheres; for instance, admitting as Z the square $[0,1] \times [0,1]$, we have the sphere $S(Z; 1/4) = \{[0,1] \times [1,2],\ [0,1] \times [-2,-1], [1,2] \times [0,1], [-2,-1] \times [0,1]\}$ etc.

A line segment may be defined via the auxiliary notion of a pattern; we introduce this notion as a functor Pt.

We let

$$Pt(X, Y, Z) \iff Z\varepsilon T_B(X, Y) \vee X\varepsilon T_B(Z, Y) \vee Y\varepsilon T_B(X, Z).$$

We will say that a finite sequence $X_1, X_2, ..., X_n$ of points belong in a line segment whenever $Pt(X_i, X_{i+1}, X_{i+2})$ for $i = 1, , ..., n-2$; formally, we introduce the functor *Line* of finite arity defined via

$Line(X_1, X_2, ..., X_n) \iff \forall i < n - 1.Pt(X_i, X_{i+1}, X_{i+2})$ and then we let $Line\_seg(X_1, X_2, ..., X_n)\varepsilon Kl(X_1, X_2, ..., X_n : Line(X_1, X_2, ..., X_n))$. In particular, classes of sequences : $([0,1] \times [i, i+1])_i$, $([i, i+1] \times [0,1])_i$, $([i, i+1] \times [i, i+1])_i$, $([i, i+1] \times [-i, -i+1])_i$ for $i \in [-n, n]$ where $n = 1, 2, \ldots$, are line segments.

It is clearly possible to introduce line segments of various types by means of specialized pattern functors. The notion of orthogonality may be introduced in a well-known way; we introduce a functor *Ortho* : for two line segments A, B, with $Z\varepsilon el(A)$, $Z\varepsilon el(B)$, we let $Ortho(A, B) \iff \exists X, Y, U, W.X, Y\varepsilon el(A) \wedge U, W\varepsilon el(B) \wedge non(X\varepsilon = (Y)) \wedge non(U\varepsilon = (W)) \wedge U\varepsilon Eq(X, Y) \wedge W\varepsilon Eq(X, Y)$ (read:A, B are orthogonal). In particular, line segments $([0,1] \times [i, i+1])_i$, $([i, i+1] \times [0,1])_i$ are orthogonal as well as are line segments $([i, i+1] \times [i, i+1])_i$, $([i, i+1] \times [-i, -i+1])_i$.

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