Chapter 16
Fundamental Mathematical Notions of the Theory of Socially Embedded Games: A Granular Computing Perspective

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Summary. The aim of this article is to present the key mathematical notions of the theory of socially embedded (or generalized) games (GGT) such as rule, rule application, and rule complex from the standpoint of granularity of information. In GGT, social actors as well as social games and, more generally, interactions, are represented by granules of information called rule complexes. Rule complexes are special cases of complexes of points of some space where the points are rules over a considered language. In many cases, rule complexes may be viewed as granules of information just because the rules constituting the rule complexes are drawn together on the base of similarity and/or functionality. On the other hand, rules represent granules of information. Therefore, rule complexes are multilevel structures representing granules of information as well. In this chapter, we also discuss mereological questions, and, in particular, we define three kinds of "crisp" ingredient-whole relationships with respect to complexes of points (viz., being of a g-element, being of a g-subset, and being of a subcomplex of a complex of points) and present some ideas on how to define the notion of a part (ingredient) to a degree in cases of complexes of points.

To my Parents Emilia and Kazimierz

1 Introduction

The theory of socially embedded (or generalized) games (GGT) [1–5] is a sociological, yet mathematically grounded framework for analyzing various forms of social games and, more generally, interactions[1] that actually may not fulfill the assumptions of classical game theory, i.e., the game theory created by von Neumann and Morgenstern [28]. In their approach, intended to formalize the economic behavior of fully rational agents, a game is specified by a collection of predetermined rules, a set of players, and possible moves and strategies for each and every player. Players

[1] Roughly speaking, a game is an interaction where the participating actors are conscious of being involved in the interaction.

have to follow the rules of the game and are not allowed to change them, unless stated otherwise. Nevertheless, they may choose a strategy and decide which of the possible moves to take.

The theory of socially embedded games extends classical game theory in several aspects: (1) Rules of the game may be underspecified, imprecise, and tolerate exceptions. (2) Participating actors (agents, players) may be neither strict rule followers nor pure rationalists maximizing a value. In fact, several pure action modalities (and a number of mixed cases) may be distinguished: instrumental rationality, normatively oriented action, procedural modality, ritual and communication, play [2]. Real actors actually may not know strategies for themselves and/or for other participants of the interaction. Nevertheless, they can modify, change, or even refuse to follow rules of the game, construct or plan actions, fabricate rules. (3) Actors' social roles and, in particular, values and norms are important factors having an impact on the behavior of the actors. (4) Information may be not only incomplete but also vague, and this imprecision is an additional source of uncertainty.

The key mathematical notions of GGT are *rule* and *complex of rules* or, in other words, *rule complex*.[2] Actors are social beings trying to realize their relationships and cultural forms. With every actor, we associate values, norms, actions and action modalities, judgment rules and algorithms, beliefs, knowledge, and roles specific to the actor. All the constituents that may be expressed in a language of representation are formalized as rules or rule complexes. In this way, we can speak of (1) value complexes consisting of evaluative rules and, in particular, norms; (2) action complexes collecting various action rules and procedures; (3) modality complexes, including rules and procedures describing possible modes of acting; (4) models formed of beliefs and knowledge; and (5) role complexes collecting all rules and complexes of rules relevant for actors' social roles in interactions. Value, action, and modality complexes as well as models are parts (more precisely, subcomplexes) of appropriate role complexes. On the other hand, a social interaction, and a game in particular, may be given the form of a rule complex as well. Such a rule complex specifies more or less precisely who the actors are; what their roles, rights, and obligations are; what the interaction is; what the action opportunities, resources, goals, procedures, and payoffs of the game are; etc. Thus, social actors, their systems as well as social interactions considered in GGT are uniformly represented by rule complexes. Rule complexes seem to be a flexible and powerful tool not only for representing actors and their interactions but also for analyzing various living problems in the area of social interactions.

Informally speaking, a rule complex is a set built of rules of a given language and/or an empty set, according to some formation rules. More precisely, it is a multilevel structure consisting of rules and other rule complexes, where a single rule or rule

[2] Primarily, rule complexes were coined by the present author as finitary multilevel structures built of rules [1]. The definition proposed in this chapter is new.

complex may occur many times at various levels. A prototype of a rule complex is an algorithm with embedded procedures. Rules occurring in the rule complex correspond to single instructions, and rule complexes occurring in it correspond to procedures of the algorithm.

The notion of granulation in the context of information, language, and reasoning was introduced by Zadeh, the founder of fuzzy sets and fuzzy logic [29], in his paper [30] from 1973. According to Zadeh's definition of granule, a *granule* is a clump of objects (or points) of some class, drawn together by indistinguishability, similarity, or functionality. Lin [9] proposes to replace the phrase "drawn together," as suggesting symmetry among objects, by the expression "drawn towards an object." It seems to be reasonable when single points may be distinguished as centers of accumulation in granules and/or where symmetry does not need to take place. We feel free to use both expressions, depending on the situation. Thus, we can say that *granulation of information* is a collection of granules (e.g., fuzzy granules, rough granules) where a granule is a clump of objects (points) drawn together and/or toward an object. Formation of granules from objects of a space (universe) is an important stage in granular computing (GC) [8–10, 17, 19, 24–26, 31–34]. As briefly characterized by Lin [9],

> "The primary goal of granular computing is to elevate the lower level data processing to a high level knowledge processing. Such an elevation is achieved by granulating the data space into a concept space. Each granule represents a certain primitive concept, and the granulation as a whole represents knowledge."

Apart from the granulation step, GC includes a representation step, consisting in naming granules and hence representing knowledge by words, and also the word-computing step.

The idea of computing with words or word computing (CW) was proposed by Zadeh in his seminal paper [32] from 1996, but the very foundations were laid many years earlier, starting with Zadeh's paper [30] from 1973. The reader interested in more recent developments is referred to [33, 34]. CW is a methodology where words are used instead of numbers for computing and reasoning. In Zadeh's original formulation, CW involves a fusion of natural languages and computation with fuzzy variables. A word is viewed as a label of a fuzzy granule. On the other hand, the granule is a denotation of the word and works as a fuzzy constraint on a variable. Under the assumption that information may be conveyed by constraining values of variables, the main role in CW is played by fuzzy constraint propagation from premises to conclusions. As pointed out by Zadeh [32],

> "There are two major imperatives for computing with words. First, computing with words is a necessity when the available information is too imprecise to justify the use of numbers. And second, when there is a tolerance for

> imprecision which can be exploited to achieve tractability, robustness, low solution costs and better rapport with reality. Exploitation of the tolerance for imprecision is an issue of central importance in CW."

Rough sets, invented by Pawlak (a good reference in English is [12]) and later developed and extended by a number of scientists (see, e.g., [6, 11, 16]), is a methodology for handling uncertainty arising from granularity of information in the domain of discourse. The form of granulation of information primarily considered in rough sets was based on indiscernibility among objects in a set [17]. The approach was later extended to capture similarity-based granulation [10, 24–26]. The very idea of rough sets is deep yet simple. Any concept that may be represented in the form of a set x of objects of a space is approximated by a pair of exact sets of objects, called the lower and upper rough approximations of x. More precisely, knowledge about a domain is represented in the form of an *information system*,[3] formalized as a pair $\mathcal{A} = (U,A)$ where U is a space of objects, A is a set of attributes, each attribute $a \in A$ is a mapping $a : U \mapsto V_a$, and V_a is the set of values of a. If an attribute $d : U \mapsto V_d$, the *decision attribute*, is added to A, $\mathcal{A}_d = (U, A \cup \{d\})$ is called a *decision system*. It is assumed that objects having identical descriptions in an information system are indiscernible to the users of the system. With each $B \subseteq A$, we can associate an equivalence relation $\mathrm{Ind}_B \subseteq U^2$, the *indiscernibility relation* on U relative to B. Intuitively, elements of the quotient structure U/Ind_B are appropriate candidates for elementary granules of information given $\mathcal{A} = (U,A)$ and $B \subseteq A$. In Polkowski and Skowron's paper [17], they are called *elementary B-pregranules* and are used to generate a Boolean algebra of B-pregranules by means of the set-theoretical operations of union, intersection, and complement. Given nonempty subsets $B_i \subseteq A$ ($i = 1,2$), by a (B_1,B_2)*-granule of information* relative to $\mathcal{A}$ and B_1, B_2, as above, we mean any pair (x_1,x_2) where x_i is a B_i-pregranule ($i = 1,2$). This construction may easily be adapted to the decision system case. The notion of granule just described was later refined and generalized [10, 24–26]. It is worth mentioning that decision rules obtained from decision systems represent granules of information.

Basic relations between granules of information are closeness (i.e., how close or similar two granules are) and inclusion (i.e., what it means that a granule x is a part or, more generally, an ingredient of a granule y). The part–whole relation is one of the main concepts of Leśniewski's mereology [7, 27]. Polkowski and Skowron extended the original system of Leśniewski to the system of rough mereology [13, 14, 18, 20] where an intuitive notion of a *part to a degree* was formalized and studied.

As mentioned above, rules in information systems represent granules of information. This observation may be generalized to arbitrary rules, as well as those considered in GGT. As built of rules, rule complexes represent compound granules of information. On the other hand, rule complexes modeling social actors and their interactions may be viewed themselves as granules of information. To see this, let us

[3] A historical name given by Pawlak, nowadays used also in other contexts.

recall that rules in such rule complexes are drawn together on the base of similarity and/or functionality. As already said, evaluative rules are collected to form value complexes; action rules and procedures are drawn together to build action complexes; beliefs and knowledge form models, etc. Role complexes, built of various rules describing actors' social roles in interactions and including the just mentioned complexes as parts, are next drawn together with some other complexes of rules (e.g., complexes of control and inference rules) to form rule complexes representing actors and their interactions.

The main theoretical issues addressed in GGT are application and transformation of rule complexes. Actors apply their rule complexes in situations of action or interaction to achieve private or group objectives, to plan and implement necessary activities, and to solve problems. Rules and rule complexes are subject to modifications or more serious transformations whenever needed and possible. For instance, application of a rule complex in a given situation s possibly changes the situation to a new situation s^*. If an actor is conscious of the transformation of s into s^*, then the actor will typically try to update his/her model of s to obtain a model of s^*. Here, the term "situation s" is a label for an imprecise concept, approximated by the actor by means of a rule complex, consisting of the actor's beliefs and knowledge about the situation s considered and called "the model of s." If we add that rules in GGT may be formulated in natural or seminatural language(s), then it becomes clear that GGT may substantially benefit by employing fuzzy-rough GC and related methods. On the other hand, GGT offers a vast area of application, hardly exploited by knowledge discovery methodologies.

In Sect. 2 of this chapter, we present a general notion of a complex of points. In Sect. 3, the notions of a g-(general)element, a g-(general) subset, and a subcomplex of a complex are defined and studied. From the mereological perspective, g-elements are proper parts, whereas g-subsets and subcomplexes are ingredients of complexes. Section 4 is devoted to the notions of a rule and a rule complex viewed from the standpoint of granularity of information. In the next section, we briefly describe the GGT model of a social actor, based on the notion of a rule complex. Section 6 contains a brief summary. Let us note that the present notions of a rule and a rule complex differ from those used in [1–5].

For any set x, we denote the cardinality of x by $\#x$, the power set by $\wp(x)$, and the complement by $-x$. The set of natural numbers (with zero) will be denoted by $\mathbf{N}$. Given sets x, x_0, y, y_0 such that $x_0 \subseteq x$ and $y_0 \subseteq y$, and a mapping $f : x \mapsto y$, by $f^{\rightarrow}(x_0)$ and $f^{\leftarrow}(y_0)$, we denote the image of x_0 and the inverse image of y_0 given by f, respectively. We abbreviate $z_0 \in z_1 \wedge z_1 \in z_2 \wedge \ldots \wedge z_n \in z_{n+1}$, where $n \in \mathbf{N}$, by $z_0 \in z_1 \in \ldots \in z_n \in z_{n+1}$.

2 The Notion of a Complex of Points

From the set-theoretical point of view, complexes of points of a space are sets, built of points of the space and/or the empty set according to some formation rules. In general, complexes of points are multilevel structures resembling algorithms with embedded procedures which, by the way, were prototypes of complexes of rules. Where points represent granules of information as in the case of rules, complexes of such points also represent some (compound) granules of information. Complexes of points can be granules of information themselves, e.g., role complexes, value complexes, action complexes. Components of these complexes are drawn together on the base of similarity and/or functionality. Thus, constituents of role complexes are linked together as they are relevant to the topics "social roles," components of value complexes are relevant to the topics "values and norms," and ingredients of action complexes are relevant to the topics "action and interaction." The notion of a complex emerged during the author's work on formalization of GGT. Primarily only complexes of rules, i.e., rule complexes were considered [1–5]. As far as application in GGT is concerned, rule complexes seem to be a convenient and sufficient tool for representing both social actors as well as their interactions in a uniform way and for analyzing various aspects of social interactions. In our opinion, the general notion of a complex of points is not only interesting in itself, but it can be useful as well.

Given a set of points U, called the space, from elements of U and the empty set, we build particular sets called complexes of points of U.

Definition 1. The class of *complexes of points of* U (or simply *complexes* if U is known from the context), written $\mathcal{C}(U)$, is the least class C of sets that is closed under the formation rules (cpl1)–(cpl4):

(cpl1) Every subset of U belongs to C.
(cpl2) If x is a set of members of C, then $\bigcup x \in C$.
(cpl3) If $x \subseteq y$ and $y \in C$, then $x \in C$ as well.
(cpl4) If $x \in C$, then the power set of x, $\wp(x)$, is a member of C.

We say that a complex x is finite if $\#x < \aleph_0$, the cardinality of $\mathbf{N}$.

Proposition 1. *Let C be a class of sets satisfying (cpl3). Then, (a) for every nonempty set x of members of C, $\bigcap x \in C$ and (b) for each $x \in C$ and a set y, $x - y \in C$.*

Hence the set-theoretical intersection of a nonempty set of complexes and the set-theoretical difference of a complex and a set are complexes.

Example 1. Let $x_i \in U$ ($i = 0, \ldots, 3$) be different points. Sets $x_4 = \{x_0, x_2\}$, $x_5 = \{x_1, x_6\}$, $x_6 = \{x_0, x_2, x_3\}$, $x_7 = \{x_0, x_1, x_4, x_5\}$, and

$$x = \{\underbrace{\{\ldots\{}_{n} x_0 \underbrace{\}\ldots\}}_{n} \mid n \in \mathbf{N}\}$$

are complexes. Unlike x, the remaining complexes are finite. x_0 occurs in x_7 three times: as an element of x_4, x_6, x_7 where x_6 is an element of x_5. On the other hand, x_0 occurs infinitely many times in x.

Given a class of sets C and a set x, let us define

$$\varphi(U,C,x) \text{ iff } \forall y \in x.(y \in U \vee y \in C) \tag{1}$$

and consider the following conditions:

(cpl5) Arbitrary sets of elements of C belong to C.
(cpl6) For every set x, $x \in C$ iff $\varphi(U,C,x)$.

Theorem 1.

(a) If C satisfies (cpl3), (cpl5), *then* (cpl4) *holds as well.*

(b) Conditions (cpl2)–(cpl4) *imply* (cpl5).

(c) Condition (cpl6) *implies* (cpli) *for* $i = 1, \ldots 5$.

(d) The class $\mathcal{C}(U)$ *satisfies* (cpl6).

Proof. We prove only (d) and leave the rest as an exercise. Let x be any set. In fact, the right-to-left part of (cpl6) holds for every class of sets C satisfying (cpl1)–(cpl4). To this end, assume that $\varphi(U,C,x)$ holds. Then $x = (x \cap U) \cup \{y \in x \mid y \in C\}$. Clearly, $x \cap U \in C$ by (cpl1). On the other hand, $\{y \in x \mid y \in C\} \in C$ by (cpl5) and (b). Hence by (cpl2), $x \in C$. Finally, notice that $\mathcal{C}(U)$ satisfies (cpl1)–(cpl4) by definition. For the left-to-right part assume that $x \in \mathcal{C}(U)$. Let us observe that the rules (cpl1)–(cpl4) are the only formation rules for building complexes of points of U. (i) If $x \subseteq U$, then for each $y \in x$, $y \in U$ as well, that is, $\varphi(U, \mathcal{C}(U), x)$ holds. (ii) Assume that (A) $x = \bigcup y$ where (B) y is a set of complexes z such that $\varphi(U, \mathcal{C}(U), z)$. Consider any $u \in x$. By (A), there is $z \in y$ such that $u \in z$. By (B), $u \in U$ or $u \in \mathcal{C}(U)$. Hence, $\varphi(U, \mathcal{C}(U), x)$. (iii) Now assume $x \subseteq y$ where $\varphi(U, \mathcal{C}(U), y)$ holds. Suppose $z \in x$. By the assumption, $z \in y$, and subsequently $z \in U$ or $z \in \mathcal{C}(U)$. Hence, $\varphi(U, \mathcal{C}(U), x)$. Finally, consider the case that (iv) $x = \wp(y)$ where $\varphi(U, \mathcal{C}(U), y)$. Let $z \in x$. By the assumption, $z \subseteq y$. Again by the assumption and (iii), $\varphi(U, \mathcal{C}(U), z)$. By virtue of the right-to-left part of (cpl6), $z \in \mathcal{C}(U)$. Hence, $\varphi(U, \mathcal{C}(U), x)$, as required. □

As a consequence, complexes may be described as follows:

Corollary 1. *A set x is a complex of points of U iff for each* $y \in x$*, it holds that* $y \in U$ *or y is a complex of points of U.*

One can easily see that $\wp(U)$ is the least class of sets satisfying (cpl1)–(cpl3). However, $\wp(U)$ is not closed under (cpl4) since $U \in \wp(U)$, but $\wp(U) \notin \wp(U)$. Thus, the conditions (cpl1)–(cpl3) alone are not sufficient to imply (cpl4). Next, one can wonder whether $\mathcal{C}(U)$ is the set of all subsets of

$$U \cup \wp(U) \cup \wp[U \cup \wp(U)] \cup \wp\{U \cup \wp(U) \cup \wp[U \cup \wp(U)]\} \ldots \quad (2)$$

It is not the case, as shown below.

Example 2. Consider $U = \{x\}$ and a complex of points of U

$$y = \{\underbrace{\{\ldots\{}_{n} x \underbrace{\}\ldots\}}_{n} \mid n \in \mathbf{N}\}.$$

The set $z = \{y\}$ is a complex of points of U as well. However, z is not a subset of the set given by (2) since $y \notin U$, $y \notin \wp(U)$, $y \notin \wp[U \cup \wp(U)]$, etc.

The class of complexes of points of U is proper, i.e., complexes of points of U do not form a set. Suppose, to the contrary, that $\mathcal{C}(U)$ is a set. By Theorem 1, $\mathcal{C}(U)$ must be a complex consisting of all complexes. Since each $x \subseteq \mathcal{C}(U)$ is a complex by (cpl3), $x \in \mathcal{C}(U)$. Hence $\wp[\mathcal{C}(U)] \subseteq \mathcal{C}(U)$. [Notice also that $\wp[\mathcal{C}(U)] \in \mathcal{C}(U)$.] Then $\#\wp[\mathcal{C}(U)] \leq \#\mathcal{C}(U)$ contrary to Cantor's theorem.

If points of U are granules of information, the corresponding complexes may be viewed as compound granules of information having a multilevel structure. The set-theoretical union and intersection of granules as well as the complement of a granule are granules of information themselves, according to [10]. We may also view a set of granules, the power set of a set of granules, pairs and, more generally, tuples of granules, and Cartesian products of sets of granules as granules of information. Finally, complexes of granules may be treated as granules of information as well.

3 Mereological Functors Associated with Complexes of Points

In this section, we discuss mereological questions in our complex-based framework. Concisely speaking, mereology is a theory of part–whole relations. This theory, invented by Leśniewski [7] in 1916, may be described after Sobociński [27] as

> "a deductive theory which inquires into the most general relations that may hold among objects [...]. Mereology can be regarded as the theory of collective classes in contradistinction to Ontology which is the theory of distributive classes."

In Leśniewski's mereology, a "part" means a proper part, i.e., it never holds that x is a part of itself. If x is a part of y or x,y are identical, then we say that x is an *ingredient of* y. Apart from elements and subsets, we distinguish three other kinds of "crisp" ingredients associated with the notion of a complex: g-elements, g-subsets, and subcomplexes of a complex. At the end of this section, we present some preliminary ideas for defining the notion of a part (or ingredient) of a complex of points to a degree.

Informally speaking, a point or complex of points x of a space U is a g-element of a complex y of points of U if x occurs in y at some level. A complex x is a g-subset of a complex y if all g-elements of x are g-elements of y as well. Thus g-subsets of y are arbitrary complexes built of g-elements of y. Subcomplexes of a complex y are also formed of g-elements of y, but the idea is different. A complex $x \neq y$ is a subcomplex of y if x can be obtained from y by deleting some occurrences of g-elements of y and possibly some parentheses. Subcomplexes of y are not g-subsets of y in general. The functor of being of a g-element of a complex, $\in_g$, is a part–whole relationship, generalizing the set-theoretical functor of being of an element of a set. The functors of being of a g-subset and being of a subcomplex of a complex, $\subseteq_g$ and $\sqsubseteq$, respectively, are ingredient–whole relationships, generalizing the set-theoretical functor of being of a subset of a set. Our aim here is not to give a formal theory in the spirit of Leśniewski's mereology but rather to investigate properties of the functors $\in_g$, $\subseteq_g$, and $\sqsubseteq$. Let us emphasize that the notion of a subcomplex is of particular interest and importance from the perspective of GGT.

Definition 2. Given a complex y of points of U, we say that x is a *g-element* of y, written $x \in_g y$, iff

$$x \in y \vee \exists n \in \mathbf{N}.\exists z_0,\ldots,z_n.x \in z_0 \in \ldots \in z_n \in y. \tag{3}$$

Observe that x is a point of U or a complex of points of U, whereas $z_0,\ldots,z_n$ must be complexes of points of U.

Let us note a few basic properties.

Proposition 2. *For every x and any complexes y,z:*

1. *If $x \in y$, then, $x \in_g y$.*
2. *If $x \in_g y$ and $y \in_g z$, then, $x \in_g z$.*
3. *If $x \in_g y$ and $y \subseteq z$, then, $x \in_g z$.*

Proof. We prove only 2. The remaining cases are left as exercises. Thus, assume that $x \in_g y$ and $y \in_g z$. The following cases hold by the definition: (i) $x \in y \in z$ or (ii) $x \in y$, and there are $n \in \mathbf{N}$ and complexes $y_0,\ldots,y_n$ such that $y \in y_0 \in \ldots \in y_n \in z$, or (iii) there are $m \in \mathbf{N}$ and complexes $x_0,\ldots,x_m$ such that $x \in x_0 \in \ldots \in x_m \in y \in z$, or (iv) there are $m,n \in \mathbf{N}$ and complexes $x_0,\ldots,x_m,y_0,\ldots,y_n$ such that $x \in x_0 \in \ldots \in x_m \in y$ and $y \in y_0 \in \ldots \in y_n \in z$. In each case, $x \in_g z$ follows by the definition. □

Two different complexes may have the same g-elements, i.e., g-elements determine complexes only in part.

Example 3. Consider $x = \{z_0, z_1\}$ and $y = \{z_1, z_2\}$ where $z_0 = \{z_2, z_3\}$, $z_1 = \{z_0, z_4\}$, and z_2, z_3, z_4 are different elements of U. Complexes x and y are different but have the same g-elements.

All g-elements of x that are points of U form the *point base* of x, $\mathrm{pb}(x)$. Similarly, all g-elements of x being complexes form the *complex base* of x, $\mathrm{cb}(x)$:

$$\begin{aligned}\mathrm{pb}(x) &\stackrel{\mathrm{def}}{=} \{y \in_g x \mid y \in U\}, \\ \mathrm{cb}(x) &\stackrel{\mathrm{def}}{=} \{y \in_g x \mid y \in \mathcal{C}(U)\}. \end{aligned} \tag{4}$$

For convenience, let us denote the set of all elements of x that are complexes by $\mathrm{cp}(x)$.

Example 4. Consider the complexes x, y from Example 3. Obviously, their point bases are equal, and the same holds for their complex bases. Thus, $\mathrm{pb}(x) = \mathrm{pb}(y) = \{z_2, z_3, z_4\}$ and $\mathrm{cb}(x) = \mathrm{cb}(y) = x$.

Example 5. Infinite complexes may have finite point bases, whereas complexes with infinite point bases may be finite. Let U be an infinite set, $x \in U$, and y be an infinite subset of U. Define $z_1 = \{\underbrace{\{\ldots\{}_{n} x \underbrace{\}\ldots\}}_{n} \mid n \in \mathbf{N}\}$ and $z_2 = \{y\}$. Then, $\mathrm{pb}(z_1) = \{x\}$ and $\mathrm{pb}(z_2) = y$.

Theorem 2. *For any complexes x, y and $\tau \in \{\mathrm{pb}, \mathrm{cb}\}$:*

1. $\mathrm{pb}(x) \cap \mathrm{cb}(x) = \emptyset$.
2. *If* $x \in \mathrm{cb}(y)$ *or* $x \subseteq y$, *then*, $\tau(x) \subseteq \tau(y)$.
3. $\mathrm{cb}(x) = \mathrm{cp}(x) \cup \bigcup\{\mathrm{cb}(y) \mid y \in \mathrm{cp}(x)\}$.
4. $\mathrm{pb}(x) = [x \cup \bigcup \mathrm{cb}(x)] \cap U$.
5. $\mathrm{cb}(x) = \emptyset$ *iff* $\mathrm{cp}(x) = \emptyset$.
6. $\mathrm{pb}(x) = \emptyset$ *iff* $x \cap U = \emptyset$ *and* $\forall y \in \mathrm{cb}(x) y \cap U = \emptyset$.
7. *If* $x \cap U = \emptyset$, *then*, $\mathrm{cb}(\bigcup x) = \bigcup\{\mathrm{cb}(y) \mid y \in \mathrm{cp}(x)\}$.
8. *If* $x \cap U = \emptyset$, *then*, $\mathrm{pb}(\bigcup x) = \mathrm{pb}(x)$.
9. $\mathrm{cb}[\wp(x)] = \wp(x) \cup \mathrm{cb}(x)$.
10. $\mathrm{pb}[\wp(x)] = \mathrm{pb}(x)$.

Proof. We prove only 3, leaving the remaining cases as exercises. To this end, assume that $z \in \mathrm{cb}(x)$ first. By definition, z is a complex such that $z \in_g x$. Hence, $z \in x$, or there are $n \in \mathbf{N}$ and complexes $z_0, \ldots, z_n$ such that $z \in z_0 \in \ldots \in z_n \in x$. In the former case, $z \in \mathrm{cp}(x)$, and we are done. In the latter, $z \in \mathrm{cb}(z_n)$ since $z \in_g z_n$ by definition. Moreover, $z_n \in \mathrm{cp}(x)$. As a consequence, $z \in \bigcup\{\mathrm{cb}(y) \mid y \in \mathrm{cp}(x)\}$. To prove the remaining part, consider a complex z such that $(*)$ $z \in x$ or $(**)$ there is a complex $y \in x$ such that $z \in \mathrm{cb}(y)$. From $(*)$, it directly follows that $z \in \mathrm{cb}(x)$. From $(**)$ and the definition, $y \in x$ and $z \in_g y$. Hence by the properties of $\in_g$, $z \in_g x$. Finally, $z \in \mathrm{cb}(x)$. □

Let us observe that complexes are well-founded sets. There is no infinite sequence of complexes $x_0, x_1, x_2, \ldots$ such that $(*) \ldots \in x_2 \in x_1 \in x_0$. By Theorem 2, the only operation responsible for an increase in the "complexity" of a complex is the power set operation. According to our definition, complexes are formed by application of (cpl1)–(cpl4) a finite, even if very large number of times. Therefore, every sequence of the form $(*)$ must be finite. There is no complex x that $x \in_g x$, either.

Recall that for any complexes (and sets in general) x and y, $x \subseteq y$ iff $\forall z.(z \in x \to z \in y)$. There arises a question what ingredient–whole relation can be obtained if we replace $\in$ by its generalization $\in_g$. We say that x is a *g-subset* of y, written $x \subseteq_g y$, iff $\forall z.(z \in_g x \to z \in_g y)$. By the definition of $\in_g$, it is easy to see that

$$x \subseteq_g y \text{ iff } \mathrm{pb}(x) \subseteq \mathrm{pb}(y) \text{ and } \mathrm{cb}(x) \subseteq \mathrm{cb}(y). \tag{5}$$

Thus, x is a g-subset of y if all of the points and complexes that form x are also points and complexes constituting y, respectively.

Example 6. Consider complexes

$$x = \{\{x_1, x_2\}, \{x_3, \{x_1, x_2\}\}\} \text{ and } y = \{x_0, \{x_3, \{x_1, x_2\}\}\}$$

where $x_i \in U$ $(i = 0, \ldots, 3)$ are different. In this case $x \subseteq_g y$ and neither $x \subseteq y$ nor $y \subseteq x$.

The fundamental properties of the notion of a g-subset are stated below.

Theorem 3. *For any complexes x, y, z:*

1. $x \subseteq_g x$.
2. *If* $x \subseteq y$, *then,* $x \subseteq_g y$.
3. *If* $x \in_g y$, *then,* $x \subseteq_g y$.
4. *If* $x \subseteq_g y$ *and* $y \subseteq_g z$, *then,* $x \subseteq_g z$.
5. *If* $x \subseteq_g y$ *and* $y \in_g z$, *then,* $x \subseteq_g z$.

Clearly, a g-subset of complex y may or may not be a g-element of y. Notice that $x \subseteq_g z$ in 5 cannot be replaced by $x \in_g z$.

Example 7. Let $x_i \in U$ $(i = 0,1,2)$ be different points, $x_3 = \{x_0\}$, $x_4 = \{x_1,x_2,x_3\}$, $x_5 = \{x_1,x_2\}$, and $x = \{x_0,x_4\}$. Observe that $x_5 \subseteq x_4$ (and hence $x_5 \subseteq_g x_4$) and $x_4 \in_g x$. However, $x_5 \notin_g x$ since $x_5 \notin \mathrm{cb}(x) = \{x_3,x_4\}$.

When complexes may be represented by collections of their g-elements, $\subseteq_g$ plays the role of $\subseteq$. Then, we would view complexes having the same g-elements as equivalent. Formally, let us define $x \equiv_g y$, where x,y are complexes, as follows:

$$x \equiv_g y \text{ iff } x \subseteq_g y \text{ and } y \subseteq_g x. \tag{6}$$

Example 8. Let $x_i \in U$ $(i = 0,\ldots,3)$ be different points. Consider two complexes $x = \{x_0,\{x_1,x_2\},\{x_3,\{x_1,x_2\}\}\}$ and $y = \{x_0,\{x_3,\{x_1,x_2\}\}\}$. Clearly, $x \equiv_g y$.

The following properties of $\equiv_g$ easily follow from Theorem 3.

Proposition 3. *For any complexes x,y,z, (a) $x \equiv_g x$; (b) $x \equiv_g y$ implies $y \equiv_g x$; and (c) if $x \equiv_g y$ and $y \equiv_g z$, then $x \equiv_g z$.*

Thus, $\mathcal{C}(U)$ is divided by $\equiv_g$ into classes of complexes with the same g-elements.

Recall briefly that a complex is a g-subset of a complex y if it is built of some (or all) g-elements of y. Another idea underlies the notion of a subcomplex. Assume that x is a complex, and consider a complex $y = \underbrace{\{\ldots\{}_{n} x \underbrace{\}\ldots\}}_{n}$ where $n \in \mathbf{N} - \{0\}$. From the GGT perspective, these parentheses are redundant[4] since application of y will resolve itself into application of x. Let us define an operation of removing parentheses, $\odot$, as follows:

$$\odot(x) \stackrel{\text{def}}{=} \begin{cases} y & \text{if } x = \{y\} \text{ and } y \in \mathcal{C}(U) \\ x & \text{otherwise.} \end{cases} \tag{7}$$

Multiple parentheses may be removed by iteration of $\odot$.

Definition 3. A complex x is a *subcomplex* of a complex y, $x \sqsubseteq y$, if $x = y$, or x may be obtained from y by deleting some occurrences of g-elements of y and/or redundant parentheses.

A few properties of subcomplexes are given below.

Theorem 4. *For any complexes x,y,z:*

1. $x \sqsubseteq x$.
2. *If* $x \subseteq y$, *then*, $x \sqsubseteq y$.
3. *If* $x \in_g y$, *then*, $x \sqsubseteq y$.
4. *If* $x \sqsubseteq y$ *and* $y \sqsubseteq z$, *then*, $x \sqsubseteq z$.
5. *If* $x \sqsubseteq y$ *and* $y \in_g z$, *then*, $x \sqsubseteq z$.

[4] This is a relatively simple redundancy. In the future, more complicated forms of redundancy should be taken into account as well.

Proof. We prove only 3 and leave the rest as an exercise. Assume that $x \in_g y$. By definition, $x \in y$, or there are $n \in \mathbf{N}$ and complexes $x_0, \ldots, x_n$ such that $x \in x_0 \in \ldots \in x_n \in y$. In the former case, $x = \odot[y - (y - \{x\})]$. Hence $x \sqsubseteq y$ by the definition. In the latter case, let $x_{-1} = x$ and $x_{n+1} = y$. Then for $i = -1, 0, \ldots, n$, $x_i = \odot[x_{i+1} - (x_{i+1} - \{x_i\})]$. Thus, x may be obtained from y according to the definition of $\sqsubseteq$, i.e., $x \sqsubseteq y$. □

As stated in 2, all subsets of a complex y are subcomplexes of y. The converse does not hold in general. Obviously a subcomplex of y may or may not be a g-element of y. The notions of g-subset and subcomplex are different as well.

Example 9. Let $x_i \in U$ $(i = 0, \ldots, 3)$ be different points, $x_4 = \{x_1\}$, $x_5 = \{x_2, x_3, x_4\}$, $x_6 = \{x_0, x_5\}$, and $y = \{x_0, x_1, x_5, x_6\}$. Define $x_7 = \{x_2, x_4\}$, $x_8 = \{x_2, x_3\}$, $x_9 = \{x_0, x_8\}$, and $x = \{x_1, x_7, x_9\}$, that is, $x_7 = x_5 - \{x_3\}$, $x_8 = x_5 - \{x_4\}$, $x_9 = (x_6 - \{x_5\}) \cup \{x_8\}$, and $x = (y - \{x_0, x_5, x_6\}) \cup \{x_7, x_9\}$. Thus, $x \sqsubseteq y$. Notice that $\mathrm{pb}(x) = \mathrm{pb}(y) = \{x_0, \ldots, x_3\}$, $\mathrm{cb}(x) = \{x_4, x_7, x_8, x_9\}$, and $\mathrm{cb}(y) = \{x_4, x_5, x_6\}$. Hence, x is neither a subset nor a g-subset of y.

Example 10. Let $x_i \in U$ $(i = 0, 1, 2)$ be different points, $x_3 = \{x_1\}$, $x_4 = \{x_2, x_3\}$, $x = \{x_3, x_4\}$, and $y = \{x_0, x_4\}$. In this case, $\mathrm{pb}(x) = \{x_1, x_2\}$, $\mathrm{pb}(y) = \{x_0, x_1, x_2\}$, and $\mathrm{cb}(x) = \mathrm{cb}(y) = \{x_3, x_4\}$. Thus, x is a g-subset of y but not a subcomplex of y.

The following observations hold by Theorem 4.

Proposition 4. *For any complexes x, z and a set y, (a) $x \cap y \sqsubseteq x$; (b) $x - y \sqsubseteq x$; (c) $x \sqsubseteq x \cup z$; (d) if $x \cap U = \emptyset$, then, $\forall z \in x. \bigcap x \sqsubseteq z$; and (e) $x \sqsubseteq \wp(x)$.*

It can be that $x \sqsubseteq y$ and $y \sqsubseteq x$, while nevertheless $x \neq y$.

Example 11. Let $z \in U$, and consider two different complexes:
$x = \{\underbrace{\{\ldots\{}_{n} z \underbrace{\}\ldots\}}_{n} \mid n \in \mathbf{N} - \{0,1\}\}$ and $y = \{\underbrace{\{\ldots\{}_{n} z \underbrace{\}\ldots\}}_{n} \mid n \in \mathbf{N} - \{0\}\}$. Clearly $x \sqsubseteq y$ since $x \subseteq y$. On the other hand, we can obtain y from x by applying $\odot$ to each and every element of x, that is, $y \sqsubseteq x$.

As for g-subsets, we may draw together complexes that are subcomplexes of one another. For this purpose, let us define $x \cong y$, where x, y are complexes, as follows:

$$x \cong y \text{ iff } x \sqsubseteq y \text{ and } y \sqsubseteq x. \tag{8}$$

The basic properties of $\cong$ follow from Theorem 4.

Proposition 5. *For any complexes x, y, z, (a) $x \cong x$; (b) if $x \cong y$, then, $y \cong x$ as well; (c) if $x \cong y$ and $y \cong z$, then, $x \cong z$.*

Thus, by means of $\cong$, the class of complexes $\mathcal{C}(U)$ is divided into classes of complexes that are subcomplexes of one another.

Polkowski and Skowron extended Leśniewski's mereology to the system of rough mereology [13,14,18,20]. The key concept of their framework is the notion of a *part to a degree*. We briefly present it here in keeping with the terminology of approximation spaces [10,23]. An *approximation space* $\mathcal{U}$ is a triple $\mathcal{U} = (U, I, \kappa)$ where U is the universe, $I : U \mapsto \wp(U)$ is an uncertainty mapping, and $\kappa : [\wp(U)]^2 \mapsto [0,1]$ is a rough inclusion function.[5] Let us assume that $u \in I(u)$ for each object $u \in U$. Hence $I^{\rightarrow}(U)$ is a covering of U. The mapping I works as a granulation function which associates with every $u \in U$ a clump of objects of U that are similar to u in some respect. Sets $I(u)$ may be called *basic granules* of information in $\mathcal{U}$. For any pair of sets of objects (x,y), the rough inclusion function κ determines the degree of inclusion of x in y. Along standard lines, we may assume that

$$\begin{aligned} &\kappa(x,y) = 1 \text{ iff } x \subseteq y, \\ &\kappa(x,y) > 0 \text{ iff } x = \emptyset \text{ or } x \cap y \neq \emptyset. \end{aligned} \tag{9}$$

Thus, x is included in y to the (highest) degree 1 iff x is a subset of y, and x is included in y to the (lowest) degree 0 iff x and y are disjoint. When x is finite, κ may be defined as follows [6]:

$$\kappa(x,y) \stackrel{\text{def}}{=} \begin{cases} \frac{\#(x\cap y)}{\#x} & \text{if } x \neq \emptyset \\ 1 & \text{otherwise.} \end{cases} \tag{10}$$

If $\kappa(x,y) = k$, then, we say that x is a *part* (or more precisely, an *ingredient*) of y *to the degree* k. Now, consider an object $u \in U$ such that $I(u)$ is finite and a set of objects $x \subseteq U$. In the crisp approach, u is an element of x or not. According to the rough approach, u may be a member of x to a degree, being a real number of the unit interval $[0,1]$. Thus, u is said to be a *part* of x *to a degree* k when $\kappa[I(u), x] = k$, i.e., if

$$\frac{\#[I(u) \cap x]}{\#I(u)} = k. \tag{11}$$

There arises a question, *how is the notion of a part to a degree defined for complexes of points?* For the time being, we are able to answer this question only partially. First, consider a complex of objects x and an object $u \in U$ such that $\#I(u) < \aleph_0$. As earlier, $\mathrm{pb}(x)$ and $\mathrm{cb}(x)$ denote the point base and the complex base of x, respectively. We say that u is a *part* of x *to a degree* $k \in [0,1]$ just in case u is a part of $\mathrm{pb}(x)$ to the degree k, with the latter notion defined as above. The definition (10) of a rough inclusion function may be extended to complexes of points. In what follows, we generalize the crisp notion of a g-subset to the rough case. Consider complexes x, y of points of U such that the point and complex bases of x are finite. We define a mapping κ_g on such pairs of complexes into $[0,1] \times [0,1]$ as follows:

$$\kappa_g(x,y) = (\kappa[\mathrm{pb}(x), \mathrm{pb}(y)], \kappa[\mathrm{cb}(x), \mathrm{cb}(y)]). \tag{12}$$

[5] For simplicity, we omit the lists of parameters occurring in the original definition.

Then, we say that x is an *ingredient* of *y to the degrees* (k_1, k_2) $(k_1, k_2 \in [0,1])$ iff $\kappa_g(x,y) = (k_1, k_2)$. According to this definition, complexes having the same point bases and complex bases are equivalent. In other words, complexes x, y such that $x \equiv_g y$ are treated as the same. In our opinion, the above definition is unfortunately not adequate for the rough version of a subcomplex.

4 Rules, Rule Complexes, and Granules of Information

A rule is a fundamental concept of GGT. Rules are major components of games and interactions. Instructions of algorithms may be seen as rules. Values and norms as well as beliefs and knowledge of social actors may also be represented in the form of rules. Let us mention action rules specifying pre- and postconditions of various actions and interactions, rules of logical inference, generative rules and specific situational rules, control rules and in particular judgment rules, strict rules and rules with exceptions, precise and vague rules, and last but not least, metarules of various kinds.

Below, we introduce a formal notion of a rule[6] over a given language $\mathcal{L}$. At the present stage, we do not specify $\mathcal{L}$ totally, assuming that, e.g., it is rich enough to formally express rules of various games considered in GGT or to mention and use the names of rules and rule complexes within $\mathcal{L}$. To denote formulas of $\mathcal{L}$, we use lowercase Greek letters with subscripts whenever needed.

By a *rule* r over $\mathcal{L}$, we mean a triple $r = (x, y, \alpha)$ where x, y are finite sets of formulas of $\mathcal{L}$ called the sets of *premises* and *justifications* of r, respectively, and α is a formula of $\mathcal{L}$ called the *conclusion* of r. Premises and justifications have to be declarative statements, whereas conclusions may be declarative or imperative but not interrogative statements. Rules without premises and justifications are called *axiomatic*. There is a one-to-one correspondence between the set of all formulas of $\mathcal{L}$ and the set of all axiomatic rules over $\mathcal{L}$. Rules without justifications are the usual if-then rules. Since a pair (a, b) is defined as the set $\{\{a\}, \{a, b\}\}$ and a triple (a, b, c) is simply $((a, b), c)$, we can easily conclude that a rule r, as above, is a complex of formulas. Thus, rule complexes over $\mathcal{L}$ are complexes of formulas of $\mathcal{L}$. The converse does not hold in general. Nevertheless, every complex of formulas may be transformed into a rule complex.

The informal meaning of $r = (x, y, \alpha)$ is that if all formulas of x hold and all formulas of y possibly hold, then one may conclude α. Thus, premises are stronger preconditions than justifications. A justification $\beta \in y$ may actually not hold, but it suffices for the sake of application of r that we do not know surely that it does not hold. The name "justification" is adopted from the formalism introduced by Reiter and widely known as *default logic* [22]. Our rules make it possible to reason by

6 The present notion of a rule differs from that used in [1,2,5].

default and to deal with exceptions that can be particularly useful when formalizing commonsense reasoning. Needless to say, such a form of reasoning is common in social life and hence, in social actions and interactions. Of course, not all rules tolerate exceptions, i.e., in many cases, the set of justifications will be empty.

In rough set methodology, decision rules obtained from decision systems represent (or more precisely, define) corresponding rough granules of information [10, 17]. Rules considered in our framework are built of formulas of $\mathcal{L}$, formulas are formed of words of $\mathcal{L}$, whereas words are labels for fuzzy granules of information (see [32]). We argue that although they are more complicated than decision rules, rules in GGT represent granules of information in the spirit of rough set methodology as well.

Let S denote a nonempty set of all situations considered. Situations that are similar and, even more, indiscernible[7] to an actor or a collective of actors form basic granules of information. More formally, let us consider an approximation space [10, 23] $\mathcal{S} = (S, I, \kappa)$ where S is the universe, $I : S \mapsto \wp(S)$ is an uncertainty mapping, and $\kappa : [\wp(S)]^2 \mapsto [0,1]$ is a rough inclusion function.[8] We assume, as earlier, that for each $s \in S$, $s \in I(s)$. Thus, $I^{\rightarrow}(S)$ is a covering of S. Basic granules of information in $\mathcal{S}$ are of the form $I(s)$ where $s \in S$. As earlier, the rough inclusion function κ satisfies (9). For finite sets of situations, κ may be defined, e.g., as in (10).

Starting with basic granules of information in $\mathcal{S}$, we can construct more compound, yet still quite simple granules of information by means of the set-theoretical operations of union, intersection, and complement. Such granules will be referred to as *simple*. We may also view sets of granules of information, the power sets of sets of granules, pairs and, more generally, tuples of granules, and Cartesian products of sets of granules as compound granules of information. Finally, complexes of granules are granules of information as well.

According to [10], a set of situations $x \subseteq S$ is *definable* in $\mathcal{S}$ if it is a union of some values of I, i.e., if there is a set $y \subseteq S$ such that $x = \bigcup I^{\rightarrow}(y)$. Consider the classical case where $I^{\rightarrow}(S)$ is a partition of S. Then an arbitrary set of situations x may be approximated by a pair of definable sets of situations

$$(\mathrm{LOW}_{\mathcal{S}}(x), \mathrm{UPP}_{\mathcal{S}}(x)), \tag{13}$$

where $\mathrm{LOW}_{\mathcal{S}}(x)$ and $\mathrm{UPP}_{\mathcal{S}}(x)$ are the *lower* and *upper rough approximations* of x in $\mathcal{S}$, respectively, defined as follows [6,12]:

$$\mathrm{LOW}_{\mathcal{S}}(x) \stackrel{\mathrm{def}}{=} \{s \in S \mid \kappa[I(s), x] = 1\},$$
$$\mathrm{UPP}_{\mathcal{S}}(x) \stackrel{\mathrm{def}}{=} \{s \in S \mid \kappa[I(s), x] > 0\}. \tag{14}$$

[7] Two situations may be treated as indiscernible not only because they cannot be distinguished. Another motivation is that the observable differences are negligible.

[8] As in the preceding section, we omit parameters for simplicity.

Thus,

$$\begin{aligned} \mathrm{LOW}_{\mathcal{S}}(x) &= \{s \in S \mid I(s) \subseteq x\}, \\ \mathrm{UPP}_{\mathcal{S}}(x) &= \{s \in S \mid I(s) \cap x \neq \emptyset\}. \end{aligned} \tag{15}$$

It is easy to prove that the lower and upper rough approximations of x may be characterized by the following equations as well:

$$\begin{aligned} \mathrm{LOW}_{\mathcal{S}}(x) &= \bigcup\{I(s) \mid s \in S \wedge I(s) \subseteq x\} \\ \mathrm{UPP}_{\mathcal{S}}(x) &= \bigcup\{I(s) \mid s \in S \wedge I(s) \cap x \neq \emptyset\}. \end{aligned} \tag{16}$$

Elements of $\mathrm{LOW}_{\mathcal{S}}(x)$ surely belong to x, whereas elements of $\mathrm{UPP}_{\mathcal{S}}(x)$ possibly belong to x, relative to $\mathcal{S}$. One can see that x is definable in $\mathcal{S}$ iff

$$\mathrm{LOW}_{\mathcal{S}}(x) = \mathrm{UPP}_{\mathcal{S}}(x). \tag{17}$$

In a more general case where $I^{\rightarrow}(S)$ is a covering but not a partition of S, the above observation is not valid. Equations (16) do not hold in general, either. The problem of finding of the best candidates for generalized lower and upper rough approximations is discussed in detail in a separate paper. Let us note only that in case I satisfies the condition

$$\forall s, s^* \in S.[s \in I(s^*) \rightarrow s^* \in I(s)], \tag{18}$$

$\mathrm{UPP}_{\mathcal{S}}(x)$ given by (14) is still a reasonable candidate for a generalized upper rough approximation of x. On the other hand,

$$\mathrm{LOW}^*{}_{\mathcal{S}}(x) = \bigcup\{I(s) \mid s \in S \wedge \forall t \in I(s).\kappa[I(t), x] = 1\} \tag{19}$$

seems to be a better candidate than $\mathrm{LOW}_{\mathcal{S}}(x)$ for a generalized lower rough approximation of x. $\mathrm{LOW}^*{}_{\mathcal{S}}(x)$ is definable in $\mathcal{S}$, whereas $\mathrm{LOW}_{\mathcal{S}}(x)$ may actually be undefinable in our sense.

With every formula α of $\mathcal{L}$, we associate the set of situations of S, where α holds. Postponing to another occasion the discussion on how to understand that a formula holds in a situation, let us define the *meaning* of α, $||\alpha||$, as the set of situations of S, where α holds. Formulas having the same meaning are semantically equivalent. If the meaning of α is definable in $\mathcal{S}$, then $||\alpha||$ is a simple granule of information in $\mathcal{S}$ represented by α. If $||\alpha||$ is not definable in $\mathcal{S}$, we may approximate it by a pair of definable sets of situations. Then α may be viewed as a symbolic representation of a pair of granules of information[9] in $\mathcal{S}$. If $I^{\rightarrow}(S)$ is a partition of S, α may be viewed as a representation of the lower and upper rough approximations of $||\alpha||$, i.e.,

$$(\mathrm{LOW}_{\mathcal{S}}(||\alpha||), \mathrm{UPP}_{\mathcal{S}}(||\alpha||)). \tag{20}$$

[9] According to the previous remarks, a pair of granules of information is a compound granule of information.

If $I^{\rightarrow}(S)$ is not a partition of S, then $\mathrm{LOW}_{\mathcal{S}}$ and $\mathrm{UPP}_{\mathcal{S}}$ should be replaced by appropriate generalizations. The meaning of a set of formulas x, $||x||$, is defined as

$$||x|| = \{||\alpha|| \mid \alpha \in x\}. \tag{21}$$

Clearly, $||x||$ is a set of granules of information in $\mathcal{S}$ and hence, may be seen as a granule of information in $\mathcal{S}$. Thus, x represents a granule of information. The meaning of a rule $r = (x,y,\alpha)$, $||r||$, may be defined as the triple $||r|| = (||x||,||y||,||\alpha||)$. Thus, $||r||$ is a granule of information in $\mathcal{S}$ as a triple of granules of information in $\mathcal{S}$, and hence, r represents a granule of information. As a consequence, transformations of rules (e.g., composition and decomposition) may be viewed as granular computing problems. Along the same lines, the meaning of a set of rules x, $||x||$, may be defined as

$$||x|| = \{||r|| \mid r \in x\}. \tag{22}$$

Arguing as before, we can draw a conclusion that x represents a granule of information. Finally, the meaning of a rule complex x, $||x||$, may be defined as

$$||x|| = \{||y|| \mid y \in x\}. \tag{23}$$

Thus, rule complexes represent compound granules of information.

Another key issue is application of rules. A necessary but usually insufficient condition for applying a rule $r = (x,y,\alpha)$ in a situation s is that r is activated in s. Given an approximation space $\mathcal{S}$, as above, we say that r is activated in a situation s if each premise of r certainly holds in s and each justification of r possibly holds in s. If $I^{\rightarrow}(S)$ is a partition of S, then r is said to be *activated* in s iff

$$\forall\beta \in x.s \in \mathrm{LOW}_{\mathcal{S}}(||\beta||) \text{ and } \forall\beta \in y.s \in \mathrm{UPP}_{\mathcal{S}}(||\beta||). \tag{24}$$

If $I^{\rightarrow}(S)$ is merely a covering of S, then $\mathrm{LOW}_{\mathcal{S}}$ and $\mathrm{UPP}_{\mathcal{S}}$ in the formula above should be replaced by appropriate generalized rough approximation mappings.

Consider situations $s \in S$ such that $I(s)$ is finite. Then we define that a formula α *holds* in a situation s *to a degree* $k \in [0,1]$ when

$$\frac{\#[I(s) \cap ||\alpha||]}{\#I(s)} = k \tag{25}$$

by a straightforward adaptation of (11), that is, α holds in a situation s to the degree k if s belongs, to the degree k, to the set of situations where α holds.

The granular computing approach sheds a new light on the problems of what we mean by "a formula holds in a situation," application of rules and rule complexes, and many others.

5 GGT Model of Social Actors in Terms of Rule Complexes

In GGT, social actors as well as the games played and, more generally, social interactions are modeled by rule complexes. These complexes represent compound granules of information, as argued in the preceding section. On the other hand, rules constituting actors' rule complexes are drawn together on the base of similarity and/or functionality, that is, such rule complexes are themselves granules of information.

In this section, we consider the actor case, but social interactions may also be represented by appropriate rule complexes. The GGT model of a social actor is presented here in a very general way. We aim more at showing the structure of the model than describing a particular actor participating in a social interaction. A more detailed presentation of GGT models of social actors and interactions will be postponed to another occasion. We do not discuss such questions as the acquisition of information/knowledge, the formation of rules and rule complexes, or communication of actors, either. With the help of the figures below, we illustrate some ideas of our modeling of social actors. The broad boundaries of the ellipses suggest the vague nature of the concepts presented, whereas the arrows point at the existence of relationships among complexes.

The totality of rules, associated with an actor i in a situation s, is organized into a rule complex called i's *actor complex* in s, written $\mathrm{ACTOR}(i,s)$. This rule complex is usually a huge granule of information, consisting of various rules and rule complexes that themselves are granules of information as well. Two main subcomplexes of $\mathrm{ACTOR}(i,s)$ are distinguished (Fig. 1): i's *role complex* in s, $\mathrm{ROLE}(i,s)$ and i's *control complex* in s, $\mathrm{CTRL}(i,s)$. The first rule complex describes the actor i's social roles in the situation s in terms of rules and rule complexes. The notion of a social role is one of the key notions of sociology. An actor may have many different and often mutually incompatible social roles in a given interaction, e.g., family roles, roles played in the workplace, the role of customer, the role of church member, etc. $\mathrm{ROLE}(i,s)$ is obtained from $\mathrm{ACTOR}(i,s)$ by neglecting all the rules of $\mathrm{ACTOR}(i,s)$ that are irrelevant to the topics "social roles." On the other hand, $\mathrm{CTRL}(i,s)$ consists of management rules and procedures that control functioning of the whole complex $\mathrm{ACTOR}(i,s)$ and its parts, describe how to derive new rules from primary ones, to draw conclusions, to make judgments, and to transform rules and rule complexes.

To play their social roles, actors are equipped with systems of norms and values, telling them what is good, bad, worth striving for, what ought to be done, and what is forbidden. In GGT, norms and values are modeled by rules and rule complexes, and hence they represent some granules of information, as argued in the preceding section. Systems of norms and values of the actor i in s are represented by a rule complex $\mathrm{VALUE}(i,s)$, referred to as i's *value complex* in s (Fig. 2). $\mathrm{VALUE}(i,s)$ is

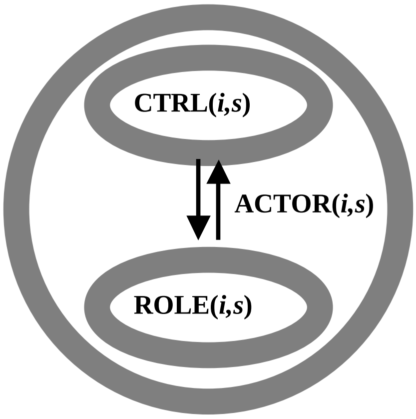

Fig. 1. An actor *i*'s rule complex and its main subcomplexes in a situation *s*

a subcomplex of ROLE(i,s). Actually, it is a granule of information obtained from the role complex by removing all rules and rule complexes irrelevant to the topics "values and norms."

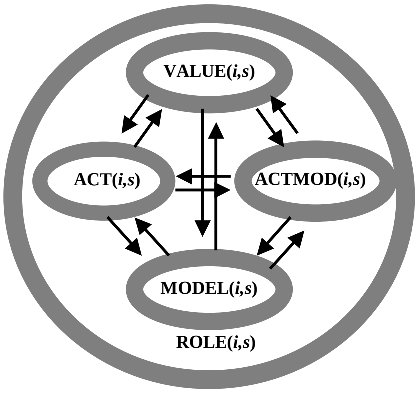

Fig. 2. Main subcomplexes of *i*'s role complex in *s*

Actors have beliefs and knowledge about themselves, about other actors involved in the interaction, and about the situation. The actor *i*'s beliefs and knowledge in *s* are represented by rules and rule complexes that form a rule complex MODEL(i,s), called *i*'s *model* in *s*. MODEL(i,s) is a granule of information, a subcomplex of ROLE(i,s) (see Fig. 2). Being to some extent provided with possible action repertoires, actors can also plan and construct appropriate actions if needed and possible. In GGT, actions are modeled by rules and rule complexes. Modes of acting are described by procedures called *action modalities*. Several pure action modalities are distinguished, viz., instrumental rationality (goal- or payoff-oriented action), normatively oriented action, procedural modality (e.g., medical procedures, bureaucratic orders), ritual and communication, and play [2]. Rules and rule complexes, representing possible actions associated with i in s, are composed into a rule complex $\mathrm{ACT}(i,s)$ called i's *action complex* in s. Action modalities of i in s are represented in the form of rule complexes that are next composed into a rule complex $\mathrm{ACTMOD}(i,s)$ referred to as i's *modality complex* in s. Both $\mathrm{ACT}(i,s)$ and $\mathrm{ACTMOD}(i,s)$ are subcomplexes of $\mathrm{ROLE}(i,s)$. They are information granules formed from $\mathrm{ROLE}(i,s)$ by neglecting all of its parts that are irrelevant to the topics "action and interaction" (Fig. 2).

Given a situation s, suppose that the actor i participates in a game or interaction G. All rules specifying and describing i's role as a player in the game G in the situation s form a subcomplex $\mathrm{ROLE}(i,s,G)$ of i's role complex $\mathrm{ROLE}(i,s)$ (see Fig. 3). Similarly, evaluative rules (i.e., rules representing values and norms) associated with G constitute a subcomplex $\mathrm{VALUE}(i,s,G)$ of i's value complex $\mathrm{VALUE}(i,s)$; action rules (i.e., rules describing actions) relevant to G form a subcomplex $\mathrm{ACT}(i,s,G)$ of $\mathrm{ACT}(i,s)$; rules and rule complexes representing action modalities appropriate for G form a subcomplex $\mathrm{ACTMOD}(i,s,G)$ of $\mathrm{ACT}(i,s)$; and finally, rules and rule complexes representing beliefs and knowledge of i in the game G in s constitute a subcomplex $\mathrm{MODEL}(i,s,G)$ of i's model $\mathrm{MODEL}(i,s)$. On the other hand, the rule complexes $\mathrm{VALUE}(i,s,G)$, $\mathrm{ACT}(i,s,G)$, $\mathrm{ACTMOD}(i,s,G)$, and $\mathrm{MODEL}(i,s,G)$ just mentioned are subcomplexes of i's role complex in the game G and the situation s, $\mathrm{ROLE}(i,s,G)$. These relationships may be depicted, as in Fig. 2, by adding a new parameter G.

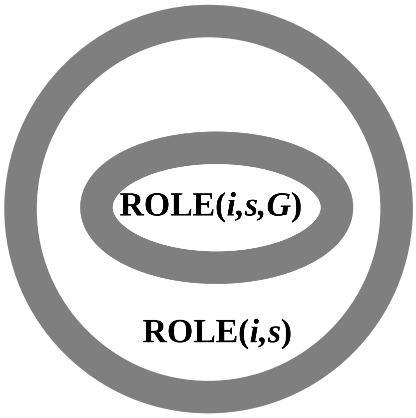

Fig. 3. An actor i's role complex relevant to a game G as a subcomplex of i's (general) role complex in s

It is hardly an exaggeration to say that, as for single rules, the main questions concerning rule complexes are application and transformation. The application of a rule complex resolves itself into the application of some rules. The transformation of a complex of points (in particular, rules) includes the transformation of points of the complex as a special case. The main, yet not necessarily elementary, kinds of transformations are composition and decomposition. Considering our previous remarks on rule complexes versus the granules of information,[10] it becomes clear that the application and transformation of rule complexes may be treated as GC problems.

6 Summary

In this chapter, we presented the mathematical foundations of the theory of socially embedded games from the perspective of granular computing. Rules in GGT represent granules of information; so do rule complexes as multilevel structures built of rules. Rule complexes are special cases of complexes of points of a space where the points are just rules over a language considered. Some rule complexes may be viewed as granules of information themselves since their constituents are drawn together on the base of similarity and/or functionality. As a consequence, the application and transformation of rules and rule complexes may be treated as GC problems. This creates new prospects for further developments of both GGT and GC. GC methods may help to solve some key problems addressed in GGT. On the other hand, those very problems may stimulate innovations and improvements of GC tools.

In this chapter, we also discussed mereological questions with respect to complexes of points. Three kinds of crisp ingredient–whole relationships were defined: being of a g-element, being of a g-subset, and being of a subcomplex of a complex of points. We presented the fundamental properties of these notions and several illustrative examples. We gave some ideas on how to define the notion of a part (ingredient) to a degree for complexes of points as well.

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[10] Rule complexes may be viewed as structures representing granules of information, and, moreover, some rule complexes are granules of information themselves.

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