Information Granules: Towards Foundations of Granular Computing

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Abstract

We introduce basic notions related to granular computing, namely the information granule syntax and semantics as well as the inclusion and closeness (similarity) relations of granules. Different information sources (units, agents) are equipped with two kinds of operations on information granules: operations possessed by agents transforming tuples of information granules definable by a given agent into information granules definable by this agent and approximation operations for computing by agents information granule approximations delivered by other agents. More complex granules are constructed by means of these operations and approximation operations from some input information granules. The construction of information granules is described by expressions called terms. We discuss a problem of synthesis of robust terms, i.e., descriptions of information granules, satisfying a given specification. This is an important problem for granular computing and its applications for spatial reasoning or knowledge discovery and data mining.

1 Introduction

A paradigm Computing with Words has been recently formulated by Lotfi Zadeh [29], [30]. Several attempts have been made to develop foundations for computing with words. Among them there is rapidly growing area of granular computing aiming to develop computing model based on information granules.

Informally speaking, information granules can be treated as linked collections (clumps) of objects drawn together by the criteria of indiscernibility, similarity or functionality [29], [30].

Notions of information granule [29], [30], [16] and information granule similarity (inclusion or closeness) are very relevant in particular for data mining and knowledge discovery in spatio-temporal databases (see e.g. [8], [20], [3], [6], [26], [19]) and also for some problems of software engineering (see, e.g. [14]).

We generalize a simple notion of granules represented by attribute value vectors as well as closeness (inclusion) relation to the case of complex information granules representing (soft) concepts [16]. We discuss examples of complex information granules. In many applications related, e.g., to knowledge discovery and data mining in spatio-temporal databases there is a need for algorithmic methods to deal with much more complex information granules than investigated so far and to discover relations between such granules. The basic relations between information granules are (rough) inclusion and closeness (similarity). The relations between information granules can be defined by extension of the relations defined on their parts. The aim is to construct a calculus in which close granules share similar relevant properties with respect to the requirement to deliver a granule satisfying a given specification (in satisfactory degree). Hence, constructions of information granules from similar ones should lead to similar complex information granules, satisfying the input specification.

The exact interpretation between granule languages of different information sources (agents) often does not exist. Hence (rough) inclusion and closeness of granules are considered instead of their equality. For example, the left and right hand sides of association rules [2] describe granules and the *support* and *confidence* coefficients specify the inclusion degree of granule represented by the formula on the left hand side into the granule represented by the formula on the right hand side of the association rule.

Reasoning in a distributed environment requires a construction of interfaces between agents (information sources or units) to enable effective learning by agents of concepts definable by other agents. The concepts definable by some agents can be not definable by the other ones. However, such concepts can be approximately definable. In the paper, we suggest a solution based on exchanging views of agents on objects with respect to a given concept. An agent delivering a concept is submitting positive and negative examples (objects) with respect to a given concept. The agent receiving this information describes objects using its own attributes. In this way a data table (called a *decision table*) is created and the approximate description of concept can be extracted by the receiving agent. Our solution is based on the rough set approach. We propose to use the parameterized approximation spaces [24, 25] to allow appropriate tuning of concept perception by agents.

An analogous method can be used in case of the *customer-agent* (agent specifying tasks) searching for a top-level cooperating agent (*root-agent*). The customer-agent presents examples and counterexamples of objects with respect to her/his concept. The concept specified by a customer-agent is approximated by agents and an agent returning the best approximation of the customer-agent concept is chosen to be the root agent. The goal of cooperating agents is to produce a concept sufficiently close to (or included in) the concept specified by the customer-agent. This concept has to be constructed from some elementary concepts available for agents called *input-agents* (inventory-agents or leaf-agents). This is realized by searching for an agent scheme [15], [17]. The schemes are represented in the paper by expressions called *terms*.

There are two main conditions to be satisfied by calculus on information granules.

- Granules should allow for compressed representation of complex nested clumps of objects (like, e.g., soft patterns) still relevant for construction of granule satisfying a given specification (in satisfactory degree). In our approach it is realized by means of information granules and constructions allowing to build more complex information granules from simpler ones together with the relevant extensions of inclusion and closeness relations.

- Concepts represented by granules should be robust with respect to their components deviations. We consider here a notion of satisfiability in a degree developed in rough mereological approach [15, 16] to deal with the problem. This together with (i) enable us to define robust granule constructions. Informally speaking, the robustness of closeness (or inclusion) under a given construction means that the closeness (inclusion) is preserved in a satisfactory degree under small deviations of input granules or operations used for the granule construction. In particular, this allows to deduce the global robustness of constructions from local robustness of their sub-constructions.

This paper contains an extension of results presented in [24, 25]. We adopt some basic ideas developed in rough mereological approach (see e.g. [15], [16], [17], [18], [10]). In our approach a crucial role play the rough set approach [13], [10] especially the approximation operations in generalized approximation spaces [21].

The paper is organized as follows. In Section 2 we recall the definition of generalized approximation space. Examples of constructions of information granules are presented in Section 3. In Section 4 their inclusion and closeness are discussed. Next, in Sections 5 and 6 the problem of information granule construction in distributed environment is investigated. In Section 7 we discuss the problem of robustness of expressions (terms) specifying granule construction. Finally, we list some problems for further investigations.

2 Rough Sets and Approximation Spaces

We recall general definition of approximation space [21], [28].

Definition 1 *A parameterized approximation space is a system* $AS_{\#,\$} = (U, I_{\#}, \nu_{\$})$, *where*

- U *is a non-empty set of objects,*

- $I_{\#} : U \to P(U)$, *where* $P(U)$ *denotes the powerset of* U, *is an uncertainty function,*

- $\nu_{\$} : P(U) \times P(U) \to [0,1]$ *is a rough inclusion function.*

The uncertainty function defines for every object x a set of similarly described objects. A constructive definition of uncertainty function can be based on the assumption that some metrics (distances) are given on attribute values. For example, if for some attribute $a \in A$ a metric $\delta_a : V_a \times V_a \longrightarrow [0,\infty)$ is given, where V_a is the set of all values of attribute a, then one can define the following uncertainty function:

$$y \in I_a^{f_a}(x) \text{ if and only if } \delta_a(a(x), a(y)) \leq f_a(a(x), a(y)),$$

where $f_a : V_a \times V_a \to [0,\infty)$ is a given threshold function.

A set $X \subseteq U$ is *definable in* $AS_{\#,\$}$, if it is a union of some values of the uncertainty function.

The rough inclusion function defines the degree of inclusion between two subsets of U [21], [16].

Thee lower and the upper approximations of subsets of U are defined as follows.

Definition 2 *For a parameterized approximation space* $AS_{\#,\$} = (U, I_{\#}, \nu_{\$})$ *and any subset* $X \subseteq U$ *the lower and the upper approximations are defined by*

$LOW(AS_{\#,\$}, X) = \{x \in U : \nu_{\$}(I_{\#}(x), X) = 1\}$,

$UPP(AS_{\#,\$}, X) = \{x \in U : \nu_{\$}(I_{\#}(x), X) > 0\}$, *respectively.*

Approximations of concepts (sets) are constructed on the basis of background knowledge. Obviously, concepts are also related to new (unseen) objects. Hence it is very useful to define parameterized approximations with parameters tuned in the searching process for approximations of concepts. This idea is crucial for construction of concept approximations using rough set methods. In our notation $\#, \$$ denote vectors of parameters which can be tuned in the process of concept approximation.

3 Syntax and Semantics of Information Granules

Usually, together with approximation space, there is also specified a set of formulas Φ expressing properties of objects. Hence, we assume that together with the approximation space $AS_{\#,\$}$ there are given

- a set of formulas Φ over some language,

- semantics $\|\cdot\|$ of formulas from Φ, i.e., a function from Φ into the power set $P(U)$.

Let us consider an example [13]. We define a language L_{IS} used for elementary granule description, where $IS = (U, A)$ is an information system. The syntax of L_{IS} is defined recursively by

1. $(a \in V) \in L_{IS}$, for any $a \in A$ and $V \subseteq V_a$.
2. If $\alpha \in L_{IS}$ then $\neg\alpha \in L_{IS}$.
3. If $\alpha, \beta \in L_{IS}$ then $\alpha \wedge \beta \in L_{IS}$.
4. If $\alpha, \beta \in L_{IS}$ then $\alpha \vee \beta \in L_{IS}$.

The semantics of formulas from L_{IS} with respect to an information system IS is defined recursively by

1. $\|a \in V\|_{IS} = \{x \in U : a(x) \in V\}$.
2. $\|\neg\alpha\|_{IS} = U - \|\alpha\|_{IS}$.
3. $\|\alpha \wedge \beta\|_{IS} = \|\alpha\|_{IS} \cap \|\beta\|_{IS}$.
4. $\|\alpha \vee \beta\|_{IS} = \|\alpha\|_{IS} \cup \|\beta\|_{IS}$.

A typical method used by the classical rough set approach [13] for constructive definition of the uncertainty function is the following: for any object $x \in U$, there is given information $Inf_A(x)$ (information signature of x relatively to A) which can be interpreted as a conjunction $EF_B(x)$ of selectors $a = a(x)$ for $a \in A$ and the set $I_{\#}(x)$ is equal to $\|EF_B(x)\|_{IS} = \left\|\bigwedge_{a\in A} a = a(x)\right\|_{IS}$. One can consider a more general case taking as possible values of $I_{\#}(x)$ any set $\|\alpha\|_{IS}$ containing x. Next from the family of such sets the resulting neighborhood $I_{\#}(x)$ can be selected or constructed. One can also use another approach by considering more general approximation spaces in which $I_{\#}(x)$ is a family of subsets of U [4], [11].

In the sequel we will consider several examples of information granule constructions. We present now the syntax and semantics of information granules. In the following section we discuss the inclusion and closeness relations for granules.

Elementary granules. In an information system $IS = (U, A)$, elementary granules are defined by $EF_B(x)$, where EF_B is a conjunction of selectors of the form $a = a(x)$, $B \subseteq A$ and $x \in U$. For example, the meaning of an elementary granule $a = 1 \wedge b = 1$ is defined by

$$\|a = 1 \wedge b = 1\|_{IS} = \{x \in U : a(x) = 1 \ \&\ b(x) = 1\}.$$

Sequences of granules. Let us assume that S is a sequence of granules and the semantics $\|\bullet\|_{IS}$ in IS of its elements have been defined. We extend $\|\bullet\|_{IS}$ on S by $\|S\|_{IS} = \{\|g\|_{IS}\}_{g\in S}$.

Example 3 *Granules defined by rules in information systems are examples of sequences of granules. Let IS be an information system and let (α, β) be a new information granule received from the rule* ***if*** *α* ***then*** *β where α, β are elementary granules of IS. The semantics $\|(\alpha, \beta)\|_{IS}$ of (α, β) is the pair of sets $(\|\alpha\|_{IS}, \|\beta\|_{IS})$.*

Sets of granules. Let us assume that a set G of granules and the semantics $\|\bullet\|_{IS}$ in IS for granules from G have been defined. We extend $\|\bullet\|_{IS}$ on the family of sets $H \subseteq G$ by $\|H\|_{IS} = \{\|g\|_{IS} : g \in H\}$.

Example 4 *One can consider granules defined by sets of rules. Assume that there is a set of rules* $Rule\_Set = \{(\alpha_i, \beta_i) : i = 1, \ldots, k\}$. *The semantics of* $Rule\_Set$ *is defined by*

$$\|Rule\_Set\|_{IS} = \{\|(\alpha_i, \beta_i)\|_{IS} : i = 1, \ldots, k\}.$$

Example 5 *One can also consider as set of granules a family of all granules* $(\alpha, Rule\_Set\,(DT_\alpha))$, *where* α *belongs to a given subset of elementary granules.*

Example 6 *Granules defined by sets of decision rules corresponding to a given evidence are also examples of sequences of granules. Let* $DT = (U, A \cup \{d\})$ *be a decision table and let* α *be an elementary granule of* $IS = (U, A)$ *such that* $\|\alpha\|_{IS} \neq \emptyset$. *Let* $Rule\_Set\,(DT_\alpha)$ *be the set of decision rules (e.g. in minimal form [10]) of the decision table* $DT_\alpha = (\|\alpha\|_{IS}, A \cup \{d\})$ *being the restriction of* DT *to objects satisfying* α. *We obtain a new granule* $(\alpha, Rule\_Set\,(DT_\alpha))$ *with the semantics*

$$\|(\alpha, Rule\_Set\,(DT_\alpha))\|_{DT} = (\|\alpha\|_{IS}, \|Rule\_Set\,(DT_\alpha)\|_{DT}).$$

This granule describes a decision algorithm applied in the situation characterized by α.

Extension of granules defined by tolerance relation. We present examples of granules obtained by application of a tolerance relation.

Example 7 *One can consider extension of elementary granules defined by tolerance relation. Let* $IS = (U, A)$ *be an information system and let* τ *be a tolerance relation on elementary granules of* IS. *Any pair* (α, τ) *is called a* τ-elementary granule. *The semantics* $\|(\alpha, \tau)\|_{IS}$ *of* (α, τ) *is the family* $\{\|\beta\|_{IS} : (\beta, \alpha) \in \tau\}$.

Example 8 *Let us consider granules defined by rules of tolerance information systems. Let* $IS = (U, A)$ *be an information system and let* τ *be a tolerance relation on elementary granules of* IS. *If* ***if*** α ***then*** β *is a rule in* IS *then the semantics of a new information granule* $(\tau : \alpha, \beta)$ *is defined by* $\|(\tau : \alpha, \beta)\|_{IS} = \|(\alpha, \tau)\|_{IS} \times \|(\beta, \tau)\|_{IS}$.

Example 9 *We consider granules defined by sets of decision rules corresponding to a given evidence in tolerance decision tables. Let* $DT = (U, A \cup \{d\})$ *be a decision table and let* τ *be a tolerance on elementary granules of* $IS = (U, A)$. *Now, any granule* $(\alpha, Rule\_Set\,(DT_\alpha))$ *can be considered as a representative of information granule cluster*

$$(\tau : (\alpha, Rule\_Set\,(DT_\alpha)))$$

with the semantics

$$\|(\tau : (\alpha, \mathit{Rule\_Set}\,(DT_\alpha)))\|_{DT} =$$

$$\left\{\|(\beta, \mathit{Rule\_Set}\,(DT_\beta))\|_{DT} : (\beta, \alpha) \in \tau\right\}.$$

Dynamic granules. An elementary granule α of the information system IS is non-empty if $\|\alpha\|_{IS} \neq \emptyset$. A non-empty elementary granule β of IS is an extension of α if $\beta = \alpha \wedge \gamma$, where γ is an elementary granule. Let us consider granules defined by some subsets of

$$\{(\beta, \mathit{Rule\_Set}\,(DT_\beta)) : \beta \text{ is an extension of } \alpha\}.$$

The semantics of these new granules is defined as in the case of sets of granules. Any set G of granules and a granule α are specifying new granules

$$\{(\beta, \mathit{Rule\_Set}\,(DT_\beta)) : \beta \text{ is an extension of } \alpha \text{ and } \beta \in G\}$$

important for decision making in dynamically changing environment. A DT-path is any sequence $\pi = ((\alpha_1, R_1), \ldots, (\alpha_k, R_k))$ such that α_i is an elementary non-empty granule of IS, $R_i = \mathit{Rule\_Set}(DT_{\alpha_i})$ for $i = 1, ..., k$ and $\alpha_i = \alpha_{i-1} \wedge \gamma_{i-1}$ for some elementary atomic granule γ_{i-1} (e.g. selector $a = v$) with an attribute $a \in A$ not appearing in α_{i-1} for $i = 2, ..., k$. A granule α_{i-1} is called a guard of π if R_{i-1} is not sufficiently close to R_i (what we denote by $\mathit{non}(\mathit{cl}_p(R_{i-1}, R_i))$, where p is the closeness degree). By $\mathit{Guard}(\pi)$ we denote the subsequence of $\alpha_1, \ldots, \alpha_k$ consisting all guards of π. In applications it is important to search for a minimal (in cardinality) granule G satisfying the following condition: for any maximal DT-path π of extensions of α all guards β from $\mathit{Guard}(\pi)$ (i.e. all points in which it is sufficient to change the decision algorithm represented by the set of decision rules) are from G. Sets of guards are symbolized in Figure 1.

One can also consider dynamic granules with tolerance relation. Let $DT = (U, A \cup \{d\})$ be a decision table and let τ be a tolerance relation on elementary granules of $IS = (U, A)$. Two DT-paths $\pi = ((\alpha_1, R_1), \ldots, (\alpha_k, R_k))$ and $\pi' = ((\beta_1, R'_1), \ldots, (\beta_l, R'_l))$ are *τ-similar* if $(\alpha_{i_s}, \beta_{j_s}) \in \tau$ for $s = 1, \ldots, r$, where $\mathit{Guard}(\pi) = (\alpha_{i_1}, \ldots, \alpha_{i_r})$ and $\mathit{Guard}(\pi') = (\beta_{j_1}, \ldots, \beta_{j_r})$. Let us assume τ has the following property:

if $(\beta, \alpha) \in \tau$ then the granules $\mathit{Rule\_Set}\,(DT_\alpha)$ and $\mathit{Rule\_Set}\,(DT_\beta)$ are sufficiently close.

We observe that having such tolerance relation one can search for a set G of guards of the smaller size. To specify the task is enough to change in the above formulated problem the condition for the maximal path to the following one: for any maximal path π of extensions of α there exists a τ-similar path π' to π such that all guards β from $\mathit{Guard}(\pi')$ (i.e. all points where it is necessary to change the decision algorithm represented by the set of decision rules) are from G.

Labeled graph granules. We discuss graph granules and labeled graph granules as notions extending previously introduced granules defined by tolerance relation and dynamic granules.

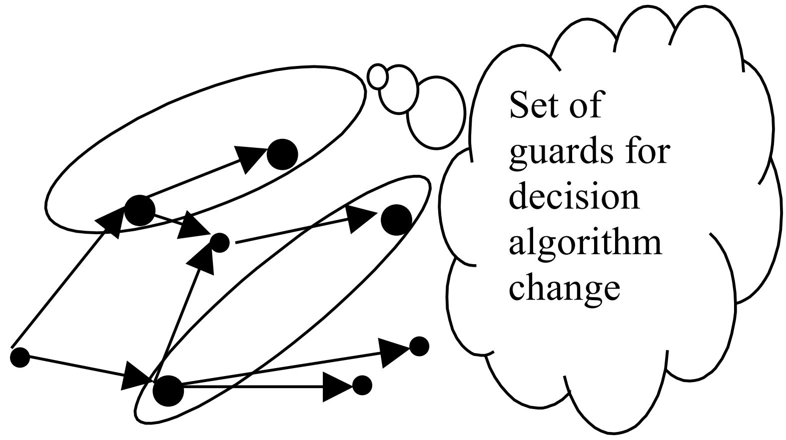

Figure 1: Two Sets of Guards

Example 10 *Let us consider granules defined by pairs* (G, E), *where* G *is a finite set of granules and* $E \subseteq G \times G$. *Let* $IS = (U, A)$ *be an information system. The semantics of a new information granule* (G, E) *is defined by* $\|(G, E)\|_{IS} = (\|G\|_{IS}, \|E\|_{IS})$, *where* $\|G\|_{IS} = \{\|g\|_{IS} : g \in G\}$ *and* $\|E\|_{IS} = \{(\|g\|, \|g'\|) : (g, g') \in E\}$.

Example 11 *Let* G *be a set of granules. Labeled graph granules over* G *are defined by* (X, E, f, h), *where* $f : X \to G$ *and* $h : E \to P(G \times G)$. *We also assume one additional condition*
if $(x, y) \in E$ *then* $(f(x), f(y)) \in h(x, y)$.
The semantics of labeled graph granule (X, E, f, h) *is defined by*

$$\{(\|f(x)\|_{IS}, \|h(x, y)\|_{IS}, \|f(y)\|_{IS}) : (x, y) \in E\}.$$

Let us summarize the above presented considerations. One can define the set of granules G as the least set containing a given set of elementary granules G_0 and closed with respect to the defined above operations of new granule construction.

We have the following examples of granule construction rules:

$$\frac{\alpha_1, \ldots, \alpha_k\text{- elementary granules}}{\{\alpha_1, \ldots, \alpha_k\}\text{- granule}}$$

$$\frac{\alpha_1, \alpha_2\text{- elementary granules}}{(\alpha_1, \alpha_2)\text{- granule}}$$

$$\frac{\alpha\text{- elementary granule }, \tau\text{- tolerance relation on elementary granules}}{(\tau : \alpha)\text{- granule}}$$

$$\frac{G\text{- a finite set of granules }, E \subseteq G \times G}{(G, E)\text{- granule}}$$

Let us observe that in case of granules constructed with application of tolerance relation we have the rule restricted to elementary granules. To obtain a more general rule like

$$\frac{\alpha\text{- graph granule }, \tau\text{- tolerance relation on graph granules}}{(\tau : \alpha)\text{- granule}}$$

it is necessary to extend the tolerance (similarity, closeness) relation on more complex objects. We discuss the problem of closeness extension in the following section.

4 Granule Inclusion and Closeness

In this section we will discuss inclusion and closeness of different information granules introduced in the previous section. Let us mention that the choice of inclusion or closeness definition depends very much on the area of application and data analyzed. This is the reason that we have decided to introduce a separate section with this more subjective part of granule semantics.

The inclusion relation between granules G, G' of degree at least p will be denoted by $\nu_p(G, G')$. Similarly, the closeness relation between granules G, G' of degree at least p will be denoted by $cl_p(G, G')$. By p we denote a vector of parameters (e.g. positive real numbers).

A general scheme for construction of hierarchical granules and their closeness can be described by the following recursive meta-rule: if granules of order $\leq k$ and their closeness have been defined then the closeness $cl_p(G, G')$ (at least in degree p) between granules G, G' of order $k+1$ can be defined by applying an appropriate operator F to closeness values of components of G, G', respectively.

A general scheme of defining more complex granule from simpler ones can be explored using rough mereological approach [15].

Inclusion and closeness of elementary granules. We have introduced the simplest case of granules in information system $IS = (U, A)$. They are defined by $EF_B(x)$, where EF_B is a conjunction of selectors of the form $a = a(x)$, $B \subseteq A$ and $x \in U$. Let $G_{IS} = \{EF_B(x) : B \subseteq A \ \& \ x \in U\}$. In the standard rough set model [13] elementary granules describe indiscernibility classes with respect to some subsets of attributes. In a more general setting see e.g. [21], [27] tolerance (similarity) classes are described.

The crisp inclusion of α in β, where $\alpha, \beta \in \{EF_B(x) : B \subseteq A \ \& \ x \in U\}$ is defined by $\|\alpha\|_{IS} \subseteq \|\beta\|_{IS}$, where $\|\alpha\|_{IS}$ and $\|\beta\|_{IS}$ are sets of objects from IS satisfying α and β, respectively. The non-crisp inclusion, known in KDD, for the case of association rules is defined by means of two thresholds t and t' :

$support_{IS}(\alpha,\beta) = card(\|\alpha \wedge \beta\|_{IS}) \geq t$, and
$accuracy_{IS}(\alpha,\beta) = \frac{support_{IS}(\alpha,\beta)}{card(\|\alpha\|_{IS})} \geq t'$.

Elementary granule inclusion in a given information system IS can be defined using different schemes, e.g., by

$\nu_{t,t'}^{IS}(\alpha,\beta)$ if and only if $support_{IS}(\alpha,\beta) \geq t$ & $accuracy_{IS}(\alpha,\beta) \geq t'$

or

$\nu_t^{IS}(\alpha,\beta)$ if and only if $accuracy_{IS}(\alpha,\beta) \geq t$.

The closeness of granules can be defined by

$cl_{t,t'}^{IS}(\alpha,\beta)$ if and only if $\nu_{t,t'}^{IS}(\alpha,\beta)$ and $\nu_{t,t'}^{IS}(\beta,\alpha)$ hold.

Decision rules as granules. One can define inclusion and closeness of granules corresponding to rules of the form **if** α **then** β using accuracy coefficients.

Having such granules $g = (\alpha,\beta)$, $g' = (\alpha',\beta')$ one can define inclusion and closeness of g and g' by $\nu_{t,t'}(g,g')$ if and only if $\nu_{t,t'}(\alpha,\alpha')$ and $\nu_{t,t'}(\beta,\beta')$.

The closeness can be defined by

$cl_{t,t'}(g,g')$ if and only if $\nu_{t,t'}(g,g')$ and $\nu_{t,t'}(g',g)$.

Another way of defining inclusion of granules corresponding to decision rules is as follows

$\nu_t^{IS}((\alpha,\beta),(\alpha',\beta'))$ if and only if $\nu_{t_1,t_2}(\alpha,\alpha')$ and $\nu_{t_1,t_2}(\beta,\beta')$ and $t = w_1 \bullet t_1 + w_2 \bullet t_2$, where w_1, w_2 are some given weights satisfying $w_1 + w_2 = 1$ and $w_1, w_2 \geq 0$.

Extensions of elementary granules by tolerance relation. For extensions of elementary granules defined by similarity (tolerance) relation, i.e., granules of the form (α,τ), (β,τ) one can consider the following inclusion measure:

$\nu_{t,t'}^{IS}((\alpha,\tau)(\beta,\tau))$ if and only if

$\nu_{t,t'}^{IS}(\alpha',\beta')$ for any α',β' such that $(\alpha,\alpha') \in \tau$ and $(\beta,\beta') \in \tau$

and the following closeness measure:

$cl_{t,t'}^{IS}((\alpha,\tau)(\beta,\tau))$ if and only if $\nu_{t,t'}^{IS}((\alpha,\tau)(\beta,\tau))$ and $\nu_{t,t'}^{IS}((\beta,\tau)(\alpha,\tau))$.

Sets of rules. It can be important for some applications to define closeness of an elementary granule α and the granule (α,τ). The definition reflecting an intuition that α should be a representation of (α,τ) sufficiently close to this granule is the following one:

$cl_{t,t'}^{IS}(\alpha,(\alpha,\tau))$ if and only if $cl_{t,t'}(\alpha,\beta)$ for any $(\alpha,\beta) \in \tau$.

An important problem related to association rules is that the number of such rules generated even from simple data table can be large. Hence, one should search for methods of aggregating close association rules. We suggest that this can be defined as searching for some close information granules.

Let us consider two finite sets $Rule\_Set$ and $Rule\_Set'$ of association rules defined by

$$Rule\_Set = \{(\alpha_i,\beta_i) : i = 1,\ldots,k\},$$

$$Rule\_Set' = \{(\alpha_i',\beta_i') : i = 1,\ldots,k'\}.$$

One can treat them as higher order information granules. These new granules $Rule\_Set$, $Rule\_Set'$ can be treated as close in a degree at least t (in IS) if and only if there exists a relation rel between sets of rules $Rule\_Set$ and $Rule\_Set'$ such that:

1. For any $Rule$ from the set $Rule\_Set$ there is $Rule'$ from $Rule\_Set'$ such that $(Rule, Rule') \in rel$ and $Rule$ is close to $Rule'$ (in IS) in degree at least t.

2. For any $Rule'$ from the set $Rule\_Set'$ there is $Rule$ from $Rule\_Set$ such that $(Rule, Rule') \in rel$ and $Rule$ is close to $Rule'$ (in IS) in degree at least t.

Another way of defining closeness of two granules G_1, G_2 represented by sets of rules can be described as follows.

Let us consider again two granules $Rule\_Set$ and $Rule\_Set'$ corresponding to two decision algorithms. By $I(\beta_i')$ we denote the set $\{j : cl_p\left(\beta_j', \beta_i'\right)\}$ for any $i = 1, \ldots, k'$.

Now, we assume $\nu_p\left(Rule\_Set, Rule\_Set'\right)$ if and only if for any $i \in \{1, \ldots, k'\}$ there exists a set $J \subseteq \{1, \ldots, k\}$ such that

$$cl_p\left(\bigvee_{j\in I\left(\beta_i'\right)} \beta_j', \bigvee_{j\in J} \beta_j\right) \text{ and } cl_p\left(\bigvee_{j\in I\left(\beta_i'\right)} \alpha_j', \bigvee_{j\in J} \alpha_j\right)$$

and for closeness we assume

$cl_p\left(Rule\_Set, Rule\_Set'\right)$ if and only if

$$\nu_p\left(Rule\_Set, Rule\_Set'\right) \text{ and } \nu_p\left(Rule\_Set', Rule\_Set\right).$$

For example if the granule G_1 consists of rules: **if** α_1 **then** $d = 1$, **if** α_2 **then** $d = 1$, **if** α_3 **then** $d = 1$, **if** β_1 **then** $d = 0$, **if** β_2 **then** $d = 0$ and the granule G_2 consists of rules: **if** γ_1 **then** $d = 1$, **if** γ_2 **then** $d = 0$, then

$cl_p\left(G_1, G_2\right)$ if and only if $cl_p\left(\alpha_1 \vee \alpha_2 \vee \alpha_3, \gamma_1\right)$ and $cl_p\left(\beta_1 \vee \beta_2, \gamma_2\right)$.

One can consider a searching problem for a granule $Rule\_Set'$ of minimal size such that $Rule\_Set$ and $Rule\_Set'$ are close (see e.g. [1]).

Granules defined by sets of granules. The previously discussed methods of inclusion and closeness definition can be easily adopted for the case of granules defined by sets of already defined granules. Let G, H be sets of granules.

The inclusion of G in H can be defined by

$\nu_{t,t'}^{IS}\left(G, H\right)$ if and only if for any $g \in G$ there is $h \in H$ for which $\nu_{t,t'}^{IS}\left(g, h\right)$ and the closeness by

$cl_{t,t'}^{IS}\left(G, H\right)$ if and only if $\nu_{t,t'}^{IS}\left(G, H\right)$ and $\nu_{t,t'}^{IS}\left(H, G\right)$.

Inclusion for complex granules specified by inclusion of their parts is symbolized in Figure 2.

Let G be a set of granules and let φ be a property of sets of granules from G (e.g. $\varphi\left(X\right)$ if and only if X is a tolerance class of a given tolerance $\tau \subseteq G \times G$). Then $P_\varphi\left(G\right) = \{X \subseteq G : \varphi\left(X\right) \text{ holds}\}$. Closeness of granules $X, Y \in P_\varphi\left(G\right)$ can be defined by

$cl_t\left(X, Y\right)$ if and only if $cl_t\left(g, g'\right)$ for any $g \in G$ and $g' \in H$.

We have the following examples of inclusion and closeness propagation rules:

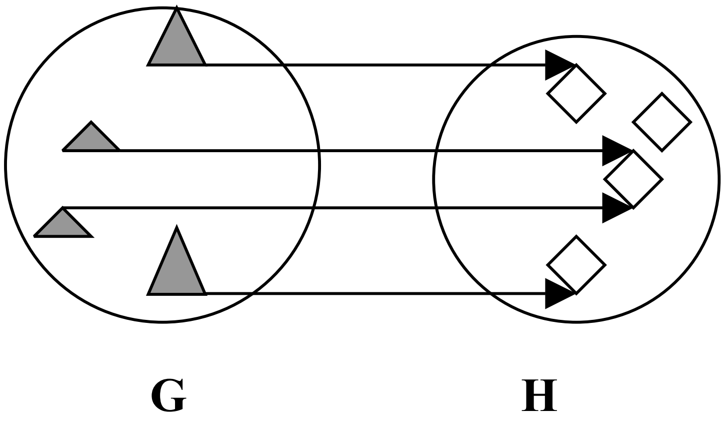

Figure 2: Two Sets of Granules

$$\frac{\text{for any } \alpha \in G \text{ there is } \alpha' \in H \text{ such that } \nu_p(\alpha, \alpha')}{\nu_p(G, H)}$$

$$\frac{cl_p(\alpha, \alpha'), cl_p(\beta, \beta')}{cl_p((\alpha, \beta), (\alpha', \beta'))}$$

$$\frac{\text{for any } \alpha' \in \tau(\alpha) \text{ there is } \beta' \in \tau(\beta) \text{ such that } \nu_p(\alpha', \beta')}{\nu_p((\tau : \alpha), (\tau : \beta))}$$

$$\frac{cl_p(G, G') \text{ and } cl_p(E, E')}{cl_p((G, E), cl_p(G', E'))}$$

where $\alpha, \alpha', \beta, \beta'$ are elementary granules, G, G' are finite sets of elementary granules.

One can also present other discussed cases for measuring the inclusion and closeness of granules in the form of inference rules. The exemplary rules have a general form, i.e., they are true in any IS (under the chosen definition of inclusion and closeness). Some of them are derivable from others. We will see in the next part of our paper that there are also some operations of new granules construction specific for a given information system. In this case one should extract from existing data these specific inference rules.

5 Mutual Understanding of Concepts by Agents

An important task for Knowledge Discovery and Data Mining (KDD) [2], [8] in distributed environment is to develop tools for modeling mutual understanding of concepts definable by different agents. Mutual understanding through communication is one of the key issues to enable collaboration among agents [9]. We assume agents specify their knowledge using data tables.

Understanding of concept definable by single agent. Let us consider two agents. There are two data tables $IS_1 = (U, A_1)$ and $IS_2 = (U, \{a\})$ corresponding to agents. We assume that $a : U \to \{0, 1\}$ is a characteristic function of a concept $X = \{x \in U : a(x) = 1\}$.

In this, typical for a rough set approach, situation the first agent is specifying the characteristic function of its concept on examples of objects. The second agent is trying to describe the concept using values of its own attributes from A_1 on objects considered by the first agent. In this way a decision table is constructed with condition attributes from A_1 and the decision a. Next, the lower and the upper approximation of the decision class X are computed. The size of the boundary region of X with respect to A_1 can be used as a measure of uncertainty in understanding X by the agent with attributes A_1.

Closeness of X to its approximations in the language used by the first agent can be represented by *accuracy of approximation*, i.e., by the coefficient $\alpha(AS_{A_1}, X) = \frac{card(LOW(AS_{A_1},X))}{card(UPP(AS_{A_1},X))}$.

The presented above approach can be used for learning by one agent of concepts definable by another agent. Let us consider again two agents. There are two data tables $IS_1 = (U, A_1)$ and $IS_2 = (U, A_2)$ corresponding to agents. We assume that in both data tables there is the same set U of objects and $A_1 = \{a_1^1, \ldots, a_l^1\}$, and $A_2 = \{a_1^2, \ldots, a_k^2\}$ are two sets of attributes, where $l > 0$ and $k > 0$ are given natural numbers. Let us consider concepts definable by attributes from the set A_2. For example suppose that we consider a concept defined by formula $(a_1^2 = 1 \wedge a_2^2 = 1) \vee a_3^2 = 1$. This is a concept definable by the second agent. Hence this agent can compute values of the characteristic function of the concept on objects from U and the first agent can find approximations of the concept following the procedure described above.

In this way we define approximations by the first agent of concepts definable by the second one.

Let us mention that the approximation operations are in general not distributive with respect to disjunction or conjunction. Hence one can not expect to construct concept approximations of the good quality from approximation of atomic concepts (e.g. selectors).

The parameterized approximation space of an agent can be used to tune up to satisfactory degree the approximation of concepts definable by another agent. Let us illustrate this idea by example.

Example 12 *We consider two agents. The second agent tries to understand a concept defined by the first agent in Table 1. The data table of the second agent*

U	d
x_1	1
x_2	1
x_3	0
x_4	1
x_5	1
x_6	0
x_7	1
x_8	0
x_9	0
x_{10}	0

Table 1: Concept Defined by First Agent

is presented in Table 3.
We consider an approximation space $AS^{A_2} = (U, I_{A_2}, \nu_{SRI})$, *where* $I_{A_2}(x) = \bigcap_{a \in A_2} I_a(x)$ *and* $y \in I_{Age}^{f_{Age}}(x)$ *if and only if*

$$diff_{Age}(Age(x), Age(y)) \leq f_{Age}(Age(x), Age(y)),$$

$$y \in I_R^{f_R}(x) \text{ if and only if } R(x) = R(y),$$

where $diff_{Age}(v, v') = \frac{|v-v'|}{\max_{Age} - \min_{Age}}$ *and* $\max_{Age}$ *and* $\min_{Age}$ *are the maximum and minimum values, respectively, for attribute Age.*

Let us consider two cases of constant threshold functions:
$f_{Age}^{0.3}(Age(x), Age(y)) = 0.3$ *and* $f_{Age}^{0.1}(Age(x), Age(y)) = 0.1$.

In the case of $f_{Age}^{0.3}$ *we obtain the following approximations:*
$LOW(AS^{A_2}, X_1) = \{x_1, x_5\}$,
$UPP(AS^{A_2}, X_1) = \{x_1, x_2, x_4, x_5, x_6, x_7, x_{10}\}$.
Thus the accuracy of approximation is less than 1.
On the other hand in the case of $f_{Age}^{0.1}$ *we obtain the following approximations:*
$LOW(AS^{A_2}, X_1) = \{x_1, x_2, x_4, x_5, x_7\}$,
$UPP(AS^{A_2}, X_1) = \{x_1, x_2, x_4, x_5, x_7\}$.
Thus the accuracy of approximation is equal to 1.

One can consider a more general case when one agent is trying to approximate some concepts using approximations of these concepts delivered by another agent. This can be used when top-down search for synthesis of concepts is performed.
There are two data tables $DT_1 = (U, A_1 \cup \{d\})$ and $DT_2 = (U, A_2 \cup \{d\})$ corresponding to agents. We assume that in both data tables there is the same set of objects U, the same decision d and $A_1 = \{a_1^1, \ldots, a_l^1\}$, and $A_2 = \{a_1^2, \ldots, a_k^2\}$ are two sets of attributes, where $l > 0$ and $k > 0$ are given natural numbers. Let

U	Sex	$Infections$	d
x_1	f	yes	1
x_2	m	yes	1
x_3	f	no	0
x_4	m	yes	1
x_5	m	yes	1
x_6	m	yes	0
x_7	f	yes	1
x_8	f	no	0
x_9	m	no	0
x_{10}	m	yes	0

Table 2: First Agent Data Table

$\{X_1, \ldots, X_r\}$ be a set of $r > 1$ decision classes defined by the decision d. For every agent one can consider a parameterized approximation space $AS^j_{\#_j,\$_j}$, where $j = 1, 2$. The question is how to tune parameters of approximation spaces such that the second agent can well approximate (understand) the approximations of decision classes obtained by the first agent (and vice versa).

The inclusion of approximations can be defined by the average accuracy coefficient:

$$\alpha_{avg}\left(AS^{A_2}_{\#_2,\$_2}, AS^{A_1}_{\#_1,\$_1}\right) =$$

$$\frac{1}{r} \bullet \sum_{i=1}^{r} \alpha\left(AS^{A_2}_{\#_2,\$_2}, LOW\left(AS^{A_1}_{\#_1,\$_1}, X_i\right)\right) +$$

$$\frac{1}{r} \bullet \sum_{i=1}^{r} \alpha\left(AS^{A_2}_{\#_2,\$_2}, UPP\left(AS^{A_1}_{\#_1,\$_1}, X_i\right)\right).$$

Example 13 *We consider two agents. The second agent tries to understand a concept approximately defined by the first agent. There are two languages L_{DT_1} and L_{DT_2} corresponding to agents and two data tables $DT_1 = (U, A_1 \cup \{d\})$ and $DT_2 = (U, A_2 \cup \{d\})$. Informally the first agent is identified by a tuple (L_{DT_1}, DT_1) and the second agent is identified by a tuple (L_{DT_2}, DT_2). We assume that in both data tables there is the same set of objects U.*

Let $DT_1 = (U, A_1 \cup \{d\})$ be a hypothetical medical data table (see Table 2) such that $U = \{x_1, \ldots, x_{10}\}$ and the set of attributes $A_1 = \{Sex, Infections\}$.

Let $DT_2 = (U, A_2 \cup \{d\})$ be a second hypothetical medical data table (see Table 3), where $A_2 = \{Age, R\}$ and R means remission.

There are two decision classes:

$X_1 = \{x \in U : d(x) = 1\} = \{x_1, x_2, x_4, x_5, x_7\}$,

$X_0 = \{x \in U : d(x) = 0\} = \{x_3, x_6, x_8, x_9, x_{10}\}$.

U	Age	R	d	$I_{Age}^{0.1}(\bullet)$	$I_R(\bullet)$
x_1	4	*no*	1	$\{x_1, x_5\}$	$\{x_1, x_3, x_8, x_9\}$
x_2	6	*yes*	1	$\{x_2, x_4, x_5, x_7\}$	$\{x_2, x_4, x_5, x_6, x_7, x_{10}\}$
x_3	8	*no*	0	$\{x_3, x_4, x_{10}\}$	$\{x_1, x_3, x_8, x_9\}$
x_4	7	*yes*	1	$\{x_2, x_3, x_4, x_7\}$	$\{x_2, x_4, x_5, x_6, x_7, x_{10}\}$
x_5	5	*yes*	1	$\{x_1, x_2, x_5, x_7\}$	$\{x_2, x_4, x_5, x_6, x_7, x_{10}\}$
x_6	10	*yes*	0	$\{x_6, x_{10}\}$	$\{x_2, x_4, x_5, x_6, x_7, x_{10}\}$
x_7	6	*yes*	1	$\{x_2, x_4, x_5, x_7\}$	$\{x_2, x_4, x_5, x_6, x_7, x_{10}\}$
x_8	13	*no*	0	$\{x_8, x_9\}$	$\{x_1, x_3, x_8, x_9\}$
x_9	14	*no*	0	$\{x_8, x_9\}$	$\{x_1, x_3, x_8, x_9\}$
x_{10}	9	*yes*	0	$\{x_3, x_6, x_{10}\}$	$\{x_2, x_4, x_5, x_6, x_7, x_{10}\}$

Table 3: Second Agent Data Table and Uncertainty Functions

We define the approximation space $AS_{l,u}^{A_1} = (U, I_{A_1}, \nu_{l,u})$ *with parameters* $0 \leq l < u \leq 1$, *where*

- $I_{A_1}(x) = \bigcap_{a \in A_1} I_a(x)$ *and* $I_a(x) = \{y \in U : a(x) = a(y)\}$ *for any* $x \in U$.
- $\nu_{l,u}(X, Y) = f_{l,u}(\nu_{SRI}(X, Y))$, *where*

$$f_{l,u}(t) = \begin{cases} 0 & \text{if} \quad t < l \\ \frac{t-l}{u-l} & \text{if} \quad l \leq t \leq u \\ 1 & \text{if} \quad t > u \end{cases}$$

and $\nu_{SRI}(X, Y) = \begin{cases} \frac{card(X \cap Y)}{card(X)} & \text{if} \quad X \neq \emptyset \\ 1 & \text{if} \quad X = \emptyset \end{cases}$ *for any* $X, Y \subseteq U$.

The lower and the upper approximations for $l = 0$ *and* $u = 1$ *are the following:*

$LOW\left(AS_{0,1}^{A_1}, X_1\right) = \{x_1, x_7\}$,

$UPP\left(AS_{0,1}^{A_1}, X_1\right) = \{x_1, x_2, x_4, x_5, x_6, x_7, x_{10}\}$,

$LOW\left(AS_{0,1}^{A_1}, X_0\right) = \{x_3, x_8, x_9\}$,

$UPP\left(AS_{0,1}^{A_1}, X_0\right) = \{x_2, x_3, x_4, x_5, x_6, x_8, x_9, x_{10}\}$.

Using other parameters one can obtain other approximations, for example for $l = 0.4$ *and* $u = 0.6$ *one can obtain:*

$LOW\left(AS_{0.4,0.6}^{A_1}, X_1\right) = \{x_1, x_2, x_4, x_5, x_6, x_7, x_{10}\}$,

$UPP\left(AS_{0.4,0.6}^{A_1}, X_1\right) = \{x_1, x_2, x_4, x_5, x_6, x_7, x_{10}\}$,

$LOW\left(AS_{0.4,0.6}^{A_1}, X_0\right) = \{x_3, x_8, x_9\}$,

$UPP\left(AS_{0.4,0.6}^{A_1}, X_0\right) = \{x_3, x_8, x_9\}$.

We consider an approximation space $AS^{A_2} = (U, I_{A_2}, \nu_{SRI})$, *where* $I_{A_2}(x) = \bigcap_{a \in A_2} I_a(x)$ *and* $y \in I_{Age}^{f_{Age}}(x)$ *if and only if*

$$diff_{Age}(Age(x), Age(y)) \le f_{Age}(Age(x), Age(y)),$$

$$y \in I_R^{f_R}(x) \text{ if and only if } R(x) = R(y),$$

where $diff_{Age}(v, v') = \frac{|v - v'|}{\max_{Age} - \min_{Age}}$ *and* $\max_{Age}$ *and* $\min_{Age}$ *are the maximum and minimum values, respectively, for attribute Age.*

A function $f_{Age}(Age(x), Age(y)) = 0.1$.

We would like to know how well the second agent is understanding the first agent. We compare the results obtained for two pairs of approximation spaces $\left(AS_{0,1}^{A_1}, AS^{A_2}\right)$ *and* $\left(AS_{0.4,0.6}^{A_1}, AS^{A_2}\right)$.

Therefore one can compute the following approximations for $l = 0$, $u = 1$ *and* X_1 :

$LOW\left(AS^{A_2}, LOW\left(AS_{0,1}^{A_1}, X_1\right)\right) = \{x_1\}$,

$UPP\left(AS^{A_2}, LOW\left(AS_{0,1}^{A_1}, X_1\right)\right) = \{x_1, x_2, x_4, x_5, x_7\}$,

$LOW\left(AS^{A_2}, UPP\left(AS_{0,1}^{A_1}, X_1\right)\right) = \{x_1, x_2, x_4, x_5, x_6, x_7, x_{10}\}$,

$UPP\left(AS^{A_2}, UPP\left(AS_{0,1}^{A_1}, X_1\right)\right) = \{x_1, x_2, x_4, x_5, x_6, x_7, x_{10}\}$,

and similarly for $l = 0.4$, $u = 0.6$ *and* X_0.

Finally we obtain $\alpha_{avg}\left(AS^{A_2}, AS_{0,1}^{A_1}\right) = \frac{31}{45}$.

On the other hand $\alpha_{avg}\left(AS^{A_2}, AS_{0.4,0.6}^{A_1}\right) = 1$.

Hence better understanding of the first agent concepts by the second agent is received in the case of parameters $l = 0.4$ *and* $u = 0.6$.

6 Rough Sets in Distributed Systems

We consider a set of agents Ag. Each agent is equipped with some approximation spaces. Agents are cooperating to solve a problem specified by a special agent called *customer-agent.* The result of cooperation is a scheme of agents. In the simplest case the scheme can be represented by a tree labeled by agents. In this tree leaves are delivering some concepts and any non-leaf agent $ag \in Ag$ is performing an operation $o(ag)$ on approximations of concepts delivered by its children. The root agent returns a concept being the result of computation by the scheme on concepts delivered by leaf agents. It is important to note that different agents use different languages. Hence concepts delivered by one agent ag_1 can be only perceived in an approximate sense by another agent ag, for illustration see Figures 3, 4 and 5.

We assume any non leaf-agent ag is equipped with an operation $o(ag) : U_{ag}^{(1)} \times \ldots \times U_{ag}^{(k)} \rightarrow U_{ag}^{(0)}$ and has different approximation spaces $AS_{ag}^{(i)} = \left(U_{ag}^{(i)}, I_{ag}^{(i)}, \nu_{SRI}\right)$, where $i = 0, \ldots, k$. We assume that the agent ag

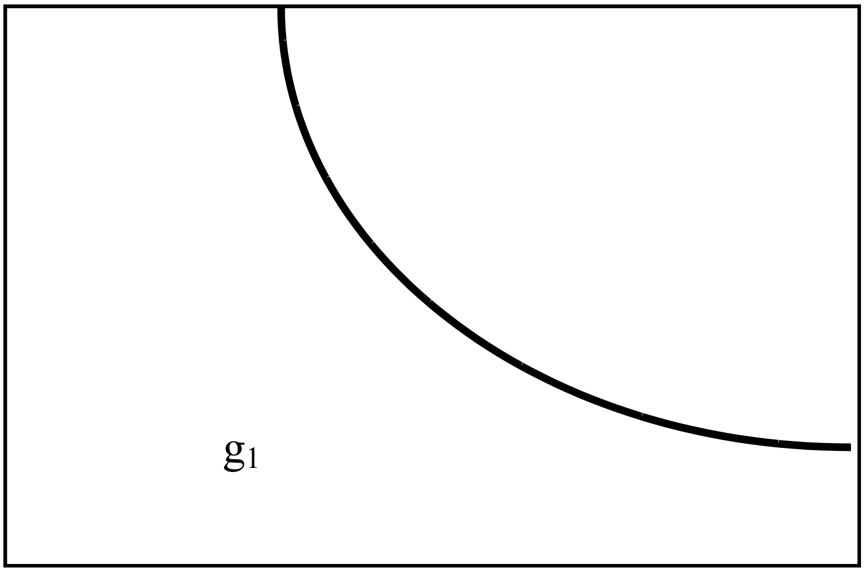

Figure 3: Concept g_1 – Information Granule of $ag_1 \in Ag$

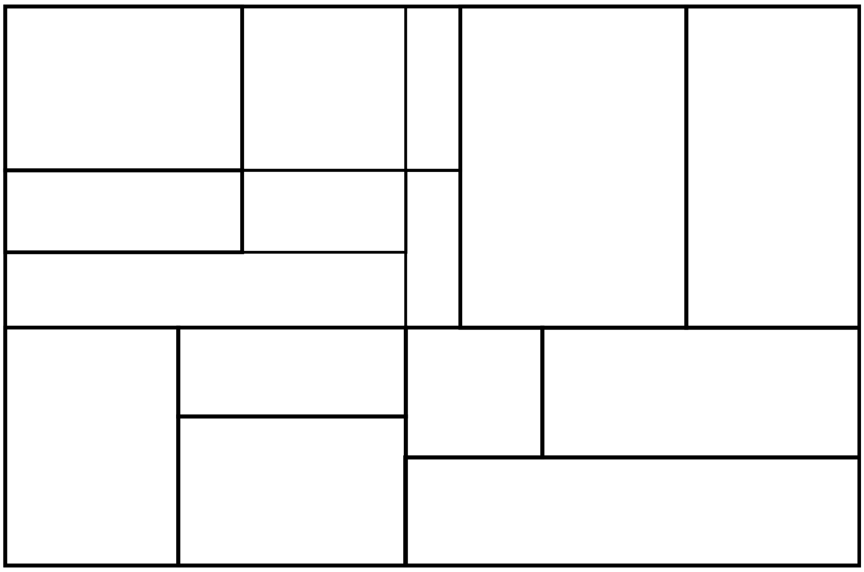

Figure 4: Communication Interface Defined by Data Table

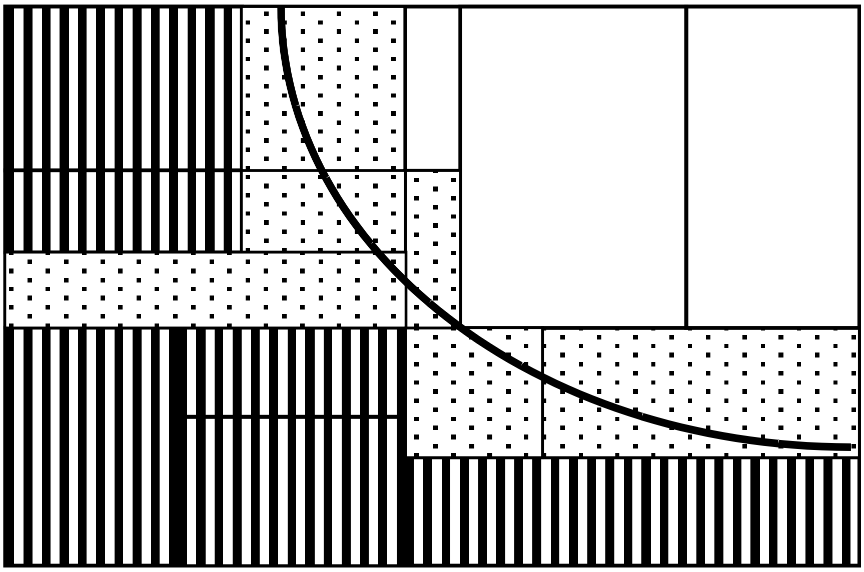

Figure 5: Lower and Upper Approximation of g_1 by $ag \in Ag$

is perceiving objects by measuring values of some available attributes. Hence some objects can become indiscernible [13]. This influences the specification of any operation $o\,(ag)$. We consider a case when arguments and values of operations are represented by attribute value vectors. Hence instead of the operation $o\,(ag)$ we have its inexact specification $o^*\,(ag)$ taking as arguments $I_{ag}^{(1)}\,(x_1)\,,\ldots,I_{ag}^{(k)}\,(x_k)$ for some $x_1 \in U_{ag}^{(1)},\ldots,x_k \in U_{ag}^{(k)}$ and returning the value $I_{ag}^{(0)}\,(o(ag)(x_1,\ldots,x_k))$ if $o(ag)(x_1,...,x_k)$ is defined, otherwise the empty set. This operation can be extended to the operation $o^*(ag)$ with arguments being definable sets (in approximation spaces attached to arguments) and with values in the family of all non-empty subsets of $U_{ag}^{(0)}$. Let $X_1,\ldots,X_k$ be definable sets. We define

$$o^*\,(ag)\,(X_1,\ldots,X_k) = \bigcup_{x_1 \in X_1,\ldots,x_k \in X_k} o^*\,(ag)\left(I_{ag}^{(1)}\,(x_1)\,,\ldots,I_{ag}^{(k)}\,(x_k)\right).$$

In the sequel, for simplicity of notation, we write $o\,(ag)$ instead of $o^*\,(ag)$.

This idea can be formalized as follows. First we define terms representing schemes of agents.

Let $X_{ag}, Y_{ag},\ldots$ be agent variables for any leaf-agent $ag \in Ag$. Let $o\,(ag)$ denote a function of arity k. We have mentioned that it is an operation from Cartesian product of $Def\_Sets(AS_{ag}^{(1)}),\ldots,Def\_Sets(AS_{ag}^{(k)})$ into $P\left(U_{ag}^{(0)}\right)$, where $Def\_Sets(AS_{ag}^{(i)})$ denotes the family of sets definable in $AS_{ag}^{(i)}$. Using the above variables and functors we define terms in a standard way, for example

$$t = o\,(ag)\,(X_{ag_1}, X_{ag_2})\,.$$

Such terms can be treated as description of complex information granules. By a valuation we mean any function val defined on the agent variables with values being definable sets satisfying $val(X_{ag}) \subseteq U_{ag}$ for any leaf-agent $ag \in Ag$. Now we can define the lower and the upper values of any term t under the valuation val with respect to a given approximation space $AS_{ag}^{(i)}$ of an agent ag

1. If t is of the form X_{ag_i} and $val(t) \subseteq U_{ag}^{(i)}$ then

$$val\left(LOW, AS_{ag}^{(i)}\right)(t) = LOW\left(AS_{ag}^{(i)}, val(t)\right)$$

$$val\left(UPP, AS_{ag}^{(i)}\right)(t) = UPP\left(AS_{ag}^{(i)}, val(t)\right)$$

   else the lower and the upper values are undefined.

2. If $t = o(ag)(t_1, \ldots, t_k)$, where $t_1, \ldots, t_k$ are terms and $o\,(ag)$ is an operation of arity k, then

   (a) if for $i = 1, \ldots, k$ $\quad val\left(LOW, AS_{ag}^{(i)}\right)(t_i)$ is defined then
   $val\left(LOW, AS_{ag}^{(0)}\right)(t) = LOW\left(AS_{ag}^{(0)}, o\,(ag)\right.$
   $\left.\left(val\left(LOW, AS_{ag}^{(1)}\right)(t_1), \ldots, val\left(LOW, AS_{ag}^{(k)}\right)(t_k)\right)\right)$
   else $val\left(LOW, AS_{ag}^{(0)}\right)(t)$ is undefined,

   (b) if for $i = 1, \ldots, k$ $\quad val\left(UPP, AS_{ag}^{(i)}\right)(t_i)$ is defined then
   $val\left(UPP, AS_{ag}^{(0)}\right)(t) = UPP\left(AS_{ag}^{(0)}, o\,(ag)\right.$
   $\left.\left(val\left(UPP, AS_{ag}^{(1)}\right)(t_1), \ldots, val\left(UPP, AS_{ag}^{(k)}\right)(t_k)\right)\right)$
   else $val\left(UPP, AS_{ag}^{(0)}\right)(t)$ is undefined.

Example 14 *Let $Ag = \{ag, ag_1, ag_2\}$ be a set of agents and let cust be an agent called customer. We explain how agents from Ag produce a granule defined by a given term and how this granule is related to the concept specified by cust. A binary operation $o(ag)$ of ag and an information system*

$$IS_{cust} = (\{w_1, w_2, w_3, w_4, w_5, w_6\}, d)$$

are described in Table 7. We assume that objects w_i, where $i = 1, \ldots, 6$ are perceived by ag using a_1^0. Two information systems IS_{ag_1}, IS_{ag_2} presented in Tables 4(a),(b) describe input information granules. Data tables $DT_1 = \left(U_{ag}^{(1)}, A_{ag}^{(1)} \cup \{d_1\}\right)$ and $DT_2 = \left(U_{ag}^{(2)}, A_{ag}^{(2)} \cup \{d_2\}\right)$ described in Table 5 and Table 6 characterize communication interfaces between agents ag_1, ag_2 and ag. A binary operation, input information granules and communication interfaces

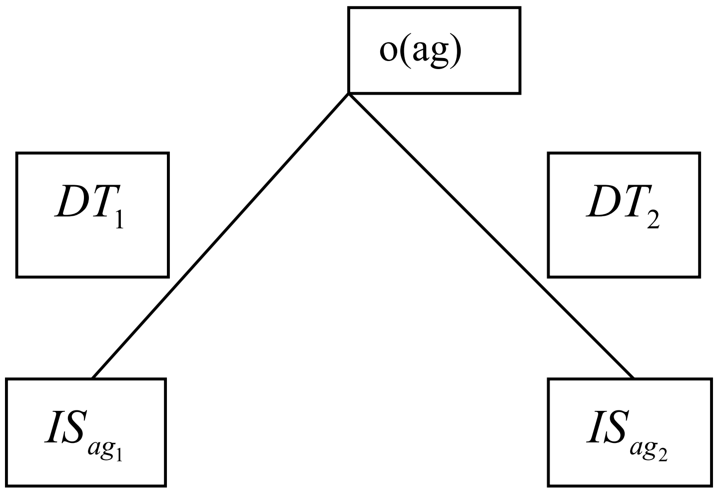

Figure 6: Operation, Input Granules and Communication Interfaces

$U^{(0)}_{ag_1}$	d_1
y_1	1
y_2	0
y_3	1
y_4	0

$U^{(0)}_{ag_2}$	d_2
z_1	1
z_2	1
z_3	1
z_4	0

Table 4: (a) Information System IS_{ag_1} (b) Information System IS_{ag_2}

$U^{(1)}_{ag}$	a^1_1	a^1_2	a^1_3	d_1	$I^{(1)}_{ag}$
y_1	yes	yes	no	1	$\{y_1\}$
y_2	no	yes	no	0	$\{y_2, y_3\}$
y_3	no	yes	no	1	$\{y_2, y_3\}$
y_4	no	no	yes	0	$\{y_4\}$

Table 5: Data Table DT_1 and Uncertainty Function $I^{(1)}_{ag}$

$U^{(2)}_{ag}$	a^2_1	a^2_2	a^2_3	d_2	$I^{(2)}_{ag}$
z_1	yes	yes	yes	1	$\{z_1, z_2\}$
z_2	yes	yes	yes	1	$\{z_1, z_2\}$
z_3	no	no	yes	1	$\{z_3, z_4\}$
z_4	no	no	yes	0	$\{z_3, z_4\}$

Table 6: Data Table DT_2 and Uncertainty Function $I^{(2)}_{ag}$

a_1^1	a_2^1	a_3^1	a_1^2	a_2^2	a_3^2	a_1^0	$U_{ag}^{(0)}$	d
yes	yes	no	yes	yes	yes	1	w_1	+
yes	yes	no	no	no	yes	2	w_2	+
no	yes	no	yes	yes	yes	3	w_3	+
no	yes	no	no	no	yes	4	w_4	-
no	no	yes	yes	yes	yes	5	w_5	-
no	no	yes	no	no	yes	6	w_6	-

Table 7: Operation $o(ag)$ and Customer Information System

are illustrated in Figure 6. The first four columns of Table 5 (6) define the information system $IS_{ag}^{(i)}$ *and the approximation space* $AS_{ag}^{(i)} = \left(U_{ag}^{(i)}, I_{ag}^{(i)}, \nu_{SRI}\right)$, *where* $i = 1, 2$.
Let $t = o(ag)(X_{ag_1}, X_{ag_2})$ *and* $val(X_{ag_1}) = \{y_1, y_3\}$. *Hence*

$$val\left(LOW, AS_{ag}^{(1)}\right)(X_{ag_1}) = LOW\left(AS_{ag}^{(1)}, \{y_1, y_3\}\right) = \{y_1\},$$
$$val\left(UPP, AS_{ag}^{(1)}\right)(X_{ag_1}) = UPP\left(AS_{ag}^{(1)}, \{y_1, y_3\}\right) = \{y_1, y_2, y_3\}.$$

Let $val(X_{ag_2}) = \{z_1, z_2, z_3\}$. *Hence*

$$val\left(LOW, AS_{ag}^{(2)}\right)(X_{ag_2}) = LOW\left(AS_{ag}^{(2)}, \{z_1, z_2, z_3\}\right) = \{z_1, z_2\},$$
$$val\left(UPP, AS_{ag}^{(2)}\right)(X_{ag_2}) = UPP\left(AS_{ag}^{(2)}, \{z_1, z_2, z_3\}\right) = \{z_1, z_2, z_3, z_4\}.$$

We obtain the lower value
$val\left(LOW, AS_{ag}^{(0)}\right)(o(ag)(X_{ag_1}, X_{ag_2})) = LOW\left(AS_{ag}^{(0)}, o(ag)\right.$
$\left.\left(val\left(LOW, AS_{ag}^{(1)}\right)(X_{ag_1}), val\left(LOW, AS_{ag}^{(2)}\right)(X_{ag_2})\right)\right) =$
$LOW\left(AS_{ag}^{(0)}, o(ag)(\{y_1\}, \{z_1, z_2\})\right) = LOW\left(AS_{ag}^{(0)}, o(ag)\right.$
$\left.\left(\left\|a_1^1 = yes \wedge a_2^1 = yes \wedge a_3^1 = no\right\|_{IS_{ag}^{(1)}}, \left\|a_1^2 = yes\right\|_{IS_{ag}^{(2)}}\right)\right) =$
$LOW\left(AS_{ag}^{(0)}, \{w_1\}\right) = \{w_1\}$.
The support of the rule

$$\textbf{\textit{if}}\ t\ \textbf{\textit{then}}\ d = +$$

under the valuation val with respect to the lower approximations is equal to

$$card\left(val\left(LOW, AS_{ag}^{(0)}\right)(t) \cap \|d = +\|_{IS_{ag}^{(0)}}\right) = 1$$

and the accuracy is also equal to 1.
We also obtain the upper value
$val\left(UPP, AS_{ag}^{(0)}\right)(o(ag)(X_{ag_1}, X_{ag_2})) = UPP\left(AS_{ag}^{(0)}, o(ag)\right.$
$\left.\left(val\left(UPP, AS_{ag}^{(1)}\right)(X_{ag_1}), val\left(UPP, AS_{ag}^{(2)}\right)(X_{ag_2})\right)\right) =$

$UPP\left(AS_{ag}^{(0)}, o(ag)(\{y_1, y_2, y_3\}, \{z_1, z_2, z_3, z_4\})\right) =$
$UPP\left(AS_{ag}^{(0)}, o(ag)\left(\|a_2^1 = yes\|_{IS_{ag}^{(1)}}, \|a_1^2 = yes \vee a_2^2 = no\|_{IS_{ag}^{(2)}}\right)\right) =$
$UPP\left(AS_{ag}^{(0)}, \{w_1, w_2, w_3, w_4\}\right) = \{w_1, w_2, w_3, w_4\}.$
*The support of the rule **if** t **then** $d = +$ under the valuation val with respect to the upper approximations is equal to*

$$card\left(val\left(UPP, AS_{ag}^{(0)}\right)(t) \cap \|d = +\|_{IS_{ag}^{(0)}}\right) = 3$$

and the accuracy is equal to 0.75.

Let us observe that the set $val(UPP, AS_{ag}^{(0)})(t) - val(LOW, AS_{ag}^{(0)})(t)$ can be treated as the boundary region of t under val. Moreover, in the process of term construction we have additional parameters to be tuned for obtaining sufficiently high support and accuracy, namely the approximation operations.

A concept X specified by the customer-agent is *sufficiently close to* t *under a given set* Val *of valuations* if X is included in the upper approximation of t under any $val \in Val$ and X includes the lower approximation of t under any $val \in Val$ as well as the size of the boundary region of t under Val, i.e.,

$$card\left(\bigcap_{val \in Val} val\left(UPP, AS_{ag}^{(0)}\right)(t) - \bigcup_{val \in Val} val\left(LOW, AS_{ag}^{(0)}\right)(t)\right),$$

is sufficiently small relatively to $\bigcap_{val \in Val} val\left(UPP, AS_{ag}^{(0)}\right)(t)$.

We conclude by formulating some examples of basic algorithmic problems.

- *Synthesis of generalized association rules.* Searching for a scheme (term t) over a given set Ag of agents and for a valuation val such that the rule **if** t **then** α, where α is a concept description specified by customer-agent, has the support at least s and the accuracy at least c under the valuation val.

- *Synthesis of concepts close to the concept specified by the customer-agent.* Searching for a scheme (term t) over a given set Ag of agents and a set Val of valuations such that the concept specified by the customer-agent is sufficiently close to t under Val and the total size of the term t and the set Val is minimal.

7 Robustness of Granules

Extracting robust patterns (i.e., patterns with properties not deviating to much under small disturbances of parameters) from data is an important task for many applications, in particular related to KDD and spatio-temporal reasoning.

We restrict our considerations for some initial remarks. Some aspects of this problem have been considered using a rough mereological framework [15], [17], [18].

Let us consider granules specified by parameterized formulas $\alpha(p)$, $\beta(p)$ together with parameterized closeness relation $cl_q(\bullet,\bullet)$ between sets of objects representing the semantics of these granules. The parameter $q \in [0,1]$ is representing the degree of closeness. We assume that values of parameter p are points in the metric space with metric ρ. Hence the intended meaning of $cl_q(\alpha(p), \beta(p))$ is that granules $\alpha(p), \beta(p)$ (parameterized by p) are close at least in the degree q. Let us assume that $\varepsilon > 0$ is a given threshold and let $cl_{q_0}(\alpha(p_0), \beta(p_0))$ holds for some p_0, q_0. We would like to check if there exists $\delta > \delta_0$ (δ_0 is a given threshold) such that for any p

if $\rho(p_0, p) \leq \delta$ then $cl_q(\alpha(p), \beta(p))$ for some q such that $q_0 - \varepsilon \leq q \leq q_0$.

The above condition is specifying that granules $\alpha(p_0), \beta(p_0)$ are not only close at least in degree q_0 but their closeness is also robust with respect to changes of parameter p.

Example 15 *Let us assume that the rule*

$$\textbf{\textit{if}}\ a \in [1, 1.5) \wedge b \in [-1, 0)\ \textbf{\textit{then}}\ d = 1$$

is an association rule in a given decision table $DT = (U, A \cup \{d\})$ with coefficients

$support_{DT}(a \in [1, 1.5) \wedge b \in [-1, 0), d = 1) = 0.6 * card(U)$,

$accuracy_{DT}(a \in [1, 1.5) \wedge b \in [-1, 0), d = 1) = 0.7$

Let $q_0 = (0.7, 0.6)$, $p_0 = (0, 0)$ and $\varepsilon = 0.1$. We ask if there exists $\delta > \delta_0 = 0.01$ such that for any $p = (p_1, p_2)$ if $\sqrt{p_1^2 + p_2^2} \leq \delta$ then the rule

$$\textbf{\textit{if}}\ a \in [1 - p_1, 1.5 + p_1) \wedge b \in [-1 - p_2, p_2)\ \textbf{\textit{then}}\ d = 1$$

is true in a given decision table DT with coefficients

$$\begin{aligned} support_1 &= support_{DT}(a \in [1 - p_1, 1.5 + p_1) \wedge \\ &\qquad b \in [-1 - p_2, p_2), d = 1), \\ accuracy_1 &= accuracy_{DT}(a \in [1 - p_1, 1.5 + p_1) \wedge \\ &\qquad b \in [-1 - p_2, p_2), d = 1), \end{aligned}$$

such that $\sqrt{\left(\frac{support_1}{card(U)} - 0.6\right)^2 + (accuracy_1 - 0.7)^2} \leq \varepsilon$.

The robustness of constructed granules can be defined by a notion analogous to the continuity of function in a given point.

Definition 16 *Let $f : X_1 \times \ldots \times X_k \to X$ and let us assume closeness relations $cl_p^{(i)}(\bullet,\bullet)$ and $cl_p(\bullet,\bullet)$ in X_i for $i = 1, \ldots, k$ and X are given. We say that f is $(\varepsilon; \varepsilon_1, ..., \varepsilon_k)$*-robust *in a given point $(x_1, \ldots, x_k)$ from $X_1 \times \ldots \times X_k$ if and only if for any $(y_1, \ldots, y_k)$ from $X_1 \times \ldots \times X_k$ if $cl_{1-\varepsilon_i}^{(i)}(x_i, y_i)$ for $i = 1, \ldots, k$ then $cl_{1-\varepsilon}(f(x_1, \ldots, x_k)), f(y_1, \ldots, y_k))$.*

We can formulate useful property for robustness checking.

Proposition 17 *Let an operation*

$$F(ag) : \prod_{i=1}^{k} Def\_Sets\left(AS_{ag}^{(i)}\right) \to Def\_Sets\left(AS_{ag}^{(0)}\right)$$

be $(\varepsilon;\varepsilon_1,\ldots,\varepsilon_k)$ *- robust in* $(g_1,\ldots,g_k)$ *and let for every agent* ag_i *operation*

$$F(ag_i) : \prod_{j=1}^{l_i} Def\_Sets\left(AS_{ag}^{(j)}\right) \rightarrow Def\_Sets\left(AS_{ag}^{(i)}\right)$$

be $(\varepsilon_i;\varepsilon_1^{(i)},\ldots,\varepsilon_{l_i}^{(i)})$ *- robust in* $(g_1^{(i)},\ldots,g_{l_i}^{(i)})$ *for* $i=1,\ldots,k$. *Then the operation*

$$F\,(ag)\,(F\,(ag_1)\,(\ldots)\,,\ldots,F\,(ag_k)\,(\ldots))$$

being the superposition of $F\,(ag)$ *and* $F\,(ag_1)\,,\ldots,F\,(ag_k)$ *is*

$$(\varepsilon;\varepsilon_1^{(1)},\ldots,\varepsilon_{l_1}^{(1)},\ldots,\varepsilon_1^{(k)},\ldots,\varepsilon_{l_k}^{(k)}) - robust$$

in

$$\left(g_1^{(1)},\ldots,g_{l_1}^{(1)},\ldots,g_1^{(k)},\ldots,g_{l_k}^{(k)}\right).$$

In Proposition 17 we consider operations with values being definable sets. These operations can be received from $o^*(ag)$ by applying the lower (or upper) approximation operation in $AS_{ag}^{(0)}$ to the arguments and values of $o^*(ag)$. Using Proposition 17 one can easily derive the robustness condition for terms describing the construction of information granules.

An important practical problem is to discover rules for decomposition of a given threshold ε into thresholds $\varepsilon_1,\ldots,\varepsilon_k$ for a given operation

$$F(ag) : \prod_{i=1}^{k} Def\_Sets\left(AS_{ag}^{(i)}\right) \rightarrow Def\_Sets\left(AS_{ag}^{(0)}\right)$$

in such a way that $F(ag)$ is $(\varepsilon;\varepsilon_1,\ldots,\varepsilon_k)$ - robust in a given point [10].

However, often a more general problem should be solved. Let us consider a set of agents Ag together with their closeness measures $cl_p^{ag}(\bullet,\bullet)$ of information granules for ag from Ag). Let $ag \in Ag$ and let g be a standard (prototype) from $Def\_Sets\left(AS_{ag}^{(0)}\right)$ together with an uncertainty coefficient $\varepsilon \in [0,1)$. Let us assume that $In(Ag)$ is a given subset of input (inventory [10]) agents from Ag. A task is to synthesize a term $t(X_{ag_1},\ldots,X_{ag_k})$, where $X_{ag_1},\ldots,X_{ag_k}$ are some variables of agents from $In(Ag)$ and a valuation val of variables such that:

- $val(LOW, AS_{ag}^{(0)})(t) \subseteq g \subseteq val(UPP, AS_{ag}^{(0)})(t)$,
- $cl_{1-\varepsilon/2}^{ag}(g, val(LOW, AS_{ag}^{(0)})(t))$ and $cl_{1-\varepsilon/2}^{ag}(g, val(UPP, AS_{ag}^{(0)})(t))$,
- the operations

$$t_{LOW}(val) = val(LOW, AS_{ag}^{(0)})(t),$$

$$t_{UPP}(val) = val(UPP, AS_{ag}^{(0)})(t)$$

defined by $t(X_{ag_1}, \ldots, X_{ag_k})$ are $(\varepsilon/2; \varepsilon_1, \ldots, \varepsilon_k)$-robust in a point

$$(val(X_{ag_1}), \ldots, val(X_{ag_k}))$$

for some $\varepsilon_1, \ldots, \varepsilon_k$ greater than a given threshold $\delta > 0$.

Now, we will discuss applications of closeness between granules for deducing that they share close sets of properties. Informally speaking, close granules should share similar properties. Assuming $\mathcal{G}$ is a given set of granules and $R \subseteq \mathcal{G} \times \mathcal{G}$ is a relation describing similarity (closeness), indiscernibility or similar functionality of information granules and $\mathcal{P}$ is a given set of properties (unary predicates) defined on information granules we can formulate the relationship between the relation R and the set $\mathcal{P}$ of properties as follows:

$$\forall g, g' \in \mathcal{G} \forall P \in \mathcal{P}[R(g, g') \rightarrow (P(g) \rightarrow P(g'))].$$

The property is called $(R, \mathcal{P})$ *-relationship*. Searching methods for relation R satisfying the specified above condition is an important problem which we will discuss in our next paper.

In case of multi-agent environment one should consider different $(R, \mathcal{P})$ -relationships for different agents. One of the fundamental question is how to deduce that close complex information granules share similar properties. Following an idea developed in rough mereological approach we can present now the main steps for such deduction.

The main role in such deduction play, so called *decomposition rules*. They have the following form:

$$o(ag) : \frac{P_{ag}(\bullet), \varepsilon, st(ag), cl_{1-\varepsilon}^{ag}(\bullet, \bullet)}{P_{ag_1}^{(1)}(\bullet), \varepsilon_1, st(ag_1), cl_{1-\varepsilon_1}^{ag_1}(\bullet, \bullet); \ldots; P_{ag_k}^{(k)}(\bullet), \varepsilon_k, st(ag_k), cl_{1-\varepsilon_k}^{ag_k}(\bullet, \bullet)}$$

where $o(ag)$ is k-ary operation of ag; $st(ag)$, $st(ag_1), \ldots, st(ag_k)$ are standard information granules (prototypes); $P_{ag}(\bullet)$, $P_{ag_1}^{(1)}(\bullet), \ldots, P_{ag_k}^{(k)}(\bullet)$ are properties; ε, $\varepsilon_1, \ldots, \varepsilon_k$ are uncertainty coefficients; $cl_{1-\varepsilon}^{ag}$, $cl_{1-\varepsilon_1}^{ag_1}, \ldots, cl_{1-\varepsilon_k}^{ag_k}$ are closeness relations attached to agents ag, $ag_1, \ldots, ag_k$, respectively.

We assume the decomposition rule is true if the following conditions are satisfied:

- $o\,(ag)\,(st\,(ag_1)\,, \ldots, st\,(ag_k)) = st\,(ag)\,,$
- $P_{ag_1}^{(1)}\,(st\,(ag_1)) \wedge \ldots \wedge P_{ag_k}^{(k)}\,(st\,(ag_k)) \rightarrow P_{ag}(st(ag)),$
- $cl_{1-\varepsilon_1}^{ag_1}(st(ag_1), g_1) \wedge \ldots \wedge cl_{1-\varepsilon_k}^{ag_k}(st(ag_k), g_k) \rightarrow cl_{1-\varepsilon}^{ag}(st(ag), g)$ for any $g_1, \ldots, g_k$ from $\mathcal{G}$, where $g = o(ag)(g_1, \ldots, g_k)$,
- $cl_{1-\varepsilon}^{ag}(g, g') \rightarrow (P_{ag}(g) \rightarrow P_{ag}(g'))$ for any $g, g' \in \mathcal{G}$.

The first condition states that the standard granule at ag is equal to the result of operation $o(ag)$ on standard granules $st(ag_1), \ldots, st(ag_k)$. The second condition allows to deduce that the standard granule $st(ag)$ has property P_{ag} if the standard granules $st(ag_1), \ldots, st(ag_k)$ have properties $P^{(1)}_{ag_1}, ..., P^{(k)}_{ag_k}$, respectively. The next condition allows to infer that $st(ag)$ is close to $o(ag)(g_1, \ldots, g_k)$ in degree at least $1 - \varepsilon$ if the standard granule $st(ag_i)$ is close to g_i in degree at least $1 - \varepsilon_i$ for any $i = 1, \ldots, k$. The last condition means that if granules g, g' are close in degree at least $1 - \varepsilon$ and g has property P_{ag} then g' has also this property.

Hence we obtain the following basic lemma:

Lemma 18 *Assuming the decomposition rule for $o(ag)$ is true and*

$$P^{(1)}_{ag_1}(st(ag_1)), cl^{ag_1}_{1-\varepsilon_1}(st(ag_1), g_1), \ldots, P^{(k)}_{ag_k}(st(ag_k)), cl^{ag_k}_{1-\varepsilon_k}(st(ag_k), g_k)$$

hold we obtain that $P_{ag}(g)$ holds, where $g = o\,(ag)\,(g_1, \ldots, g_k)$.

One of the challenges for knowledge discovery is to discover decomposition rules for operations possessed by agents [12]. The details of this problem will be presented in our next paper.

It is easy to observe that by repeated application of decomposition rules one can derive sufficient conditions for synthesis of robust complex information granules.

Conclusions

Our approach can be viewed as a step towards the understanding of complex information granules and their role in application areas like spatial reasoning or data mining and knowledge discovery. Methods for synthesis of complex information granules in distributed environment are important in these areas because of need for approximate fusion (merging) of information from different sources of information. We have discussed information granule syntax and semantics as well their inclusion and closeness. Several examples of information granules have been presented. An approach for synthesis and analysis of complex information granules in distributed environment has been outlined. It was emphasized the necessity of robust information granules synthesis. The robustness guarantees that a properly constructed scheme of reasoning on prototypes (standards) leads to conclusion that a given specification is also satisfied (in satisfactory degree) when the standards are substituted by sufficiently small deviations of them.

There is still much work to be done to create foundations of granular computing. Among the issues to be investigated are:

(i) discovery methods for decomposition rules;

(ii) synthesis from data of rules for uncertainty coefficient propagation;

(iii) efficient identification of relevant fragments of synthesized schemes of reasoning to be used in further solution synthesis;

(iv) methods for fusion operation discovery;

(v) adaptive spatial reasoning.

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