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Rough Mereology in Information Systems with Applications to Qualitative Spatial Reasoning

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Abstract. *Rough Mereology* has been proposed as a paradigm for approximate reasoning in complex information systems. Its primitive notion is that of a predicate of *rough inclusion* which gives for any two entities of discourse the degree in which one of them is a part of the other. Rough Mereology may be regarded as an extension of Rough Set Theory as it proposes to argue in terms of similarity relations induced from a rough inclusion instead of reasoning in terms of more strict indiscernibility relations.

Rough Mereology is also a generalization of Mereology i.e. a theory of reasoning based on the notion of a *part*. Classical languages of mathematics are of two–fold kind: the language of set theory (naive or formal) expressing classes of objects as sets consisting of "elements", "points" etc. suitable for objects perceived as built of "atoms" and applied to structures perceived as discrete and the language of part relations suitable for e.g. continuous objects like solids, regions, etc. where two objects are related to each other by saying that one of them is a part of the other. Mereological theories for reasoning about complex structures are at the heart of Qualitative Spatial Reasoning.

In this paper, we study basic aspects of Rough Mereology in Information Systems. Mereology makes the distinction between entities perceived as individuals (singletons), to which the part predicate may be applied, and entities perceived as distributive classes (sets, lists, general names etc.) of entities. This distinction is made formal and precise within Ontology i.e. Theory of Being based on the primitive notion of the copula *is* which is also a basic ingredient of theories for Spatial Reasoning. The practical aim of Ontology is to elaborate a system of concepts (notions, names, sets of entities) about which the reasoning is carried out.

Therefore, we begin our study with an analysis of a simple rough set–based Ontology (the *template ontology*)in Information Systems and in this setting we present our approach to Mereology in Information Systems.

In this framework we introduce Rough Mereology and we present some ways for defining rough inclusions. We demonstrate applications of Rough Mereology to approximate reasoning taking as the case subject Qualitative Spatial Reasoning. We address here some of its mereo–topological as well as mereo–geometrical aspects.

Keywords: rough sets, information systems/tables, ontology, mereology, rough mereology, spatial reasoning, mereo–topology, mereo–geometry

1. Introduction

Rough Mereology has been proposed as a tool for reasoning under uncertainty (approximate reasoning) with data collected in information systems as well as a general paradigm within which it would be possible to formally describe schemes for synthesis of approximate solutions to problems posed uncertainly, vaguely or incompletely [64], [65], [66], [67], [75]. It has been shown to constitute a general framework (cf. Chapter 3: A Perspective, in ([68], vol.1) in which it is possible to develop a theory of rough computation with applications to distributed computing, knowledge granulation and computing with words (cf. [98], [99]).

Two guiding paradigms for Rough Mereology were: the Theory of Rough Sets [61] and the Theory of Fuzzy Sets [100]. From Rough Set Theory inherits Rough Mereology the idea of approximation of general concepts with particular i.e. exact concepts: in Rough Set Theory exact concepts are defined by means of attribute–value descriptors and a fortiori they are finite unions of indiscernibility classes with the indiscernibility relations providing partitions of the universe of objects. In recent investigations the need for more relaxed approximations, preferably induced by similarity or tolerance relations has been stressed and experimentally verified to yield better classification results [78], [74], [87], [73], [8], [40].

Rough Mereology proposes as its primitive notion the notion of a *rough inclusion* which is a parameterized predicate μ_r such that for any pair of individual entities X, Y the formula $Y \varepsilon \mu_r(X)$ means that Y is a *part of* X *to a degree* r where $r \in [0,1]$.

The rough inclusion may be regarded thus as a parameterized family of similarity relations. One may then define approximations of concepts using the family $\{\mu_r : r \in [0,1]\}$ along the lines of Rough Set Theory with modifications described e.g. in [87].

As any rough inclusion is concerned with relations among objects expressed by means of degrees of partial containment of objects, Rough Mereology has clear connections to Fuzzy Set Theory whose basic subject of study are partial containment as well as partial membership cf. [100].

Yet another source of ideas and points of reference for Rough Mereology are Mereological Theories of Sets. We refer here to two mainstream theories of Mereology viz. Mereology due to Stanisław Leśniewski [51], [53], [54], [83], [84], [20], [89], [15], [50] and Mereology based on Connection [14], [48], [95], [17], [18], [57], [4], [6].

Of the two theories, Mereology based on Connection offers a richer variety of mereo–topological functors; yet, as Mereology of Leśniewski is based on the notion of *part*, it offers a formalism of which the formalism of Rough Mereology may be – under a suitable choice of primitive expressions – a direct extension and generalization. This is actually the case: Rough Mereology was proposed [64], [65], [66], [67] to contain Mereology of Leśniewski as the theory of the predicate μ_1 and this feature is preserved in the formalization proposed here.

In the scheme envisioned by Stanisław Leśniewski, Mereology was to follow Ontology i.e. the Theory of Names (Concepts) and with this purpose in mind he proposed his Ontology [37], [49], [52], [79]. Ontological theories play an important role in Approximate Reasoning [12], [34], [80] witnessed with particular clearness in Spatial Reasoning [56], [24] where Ontology plays a basic role as it sets spatial concepts and their taxonomy; for the same reason Ontology is an immanent, although frequently implicit, component of Rough Set Theory as the latter discusses concepts and their approximations hence it is vitally interested in taxonomies of concepts and relations among them.

Mereology may be applied in exact schemes of reasoning (similarly to logic on which exact schemes of reasoning are based) related to objects whose structure (in terms of their decomposition into parts) is well–known; however, when our reasoning is applied to situations where our knowledge is uncertain, incomplete etc. we may outline a decomposition scheme only approximatively i.e. we may evaluate only that our object in question is composed of some other objects as parts up to a certain degree. In consequence, other constructs used in spatial reasoning like a neighborhood, an interior, etc. are defined approximatively only, the degree of approximation determined in an intricate way by degrees of being a part.

From this place a two–fold way stretches forth; first, we may localize Rough Mereology within Ontology in particular within Ontology of Information Systems (i.e. Rough Set Ontology) and next, we may use Rough Mereology to create Ontology for a particular domain of interest. Here we select as such Case Study domain the domain of Spatial Reasoning because of the eminent role played in its development by mereology–based methods.

The two ways meet again in the final sections where we discuss rough mereo–topology and rough mereo–geometry in the context of Spatial Reasoning in Information Systems.

The purpose of mereo–topology is to express approximate topological relations among objects, while mereo–geometry attempts at description of geometric features of sets of objects, like *betweenness*, *orthogonality* etc. again, approximately only. In these approximations, we exploit rough inclusions regarded as weak metrics.

Let us observe here in passing that rough set–theoretic ideas have already been applied in problems of Spatial Reasoning e.g. in problems of multi–resolution spatial reasoning [86], [97], in problems of localization (rough localization) [10], and in the *egg–yolk approach* [19], where a region with uncertain boundary is enclosed in two regions with definite boundaries, one may find ideas very close to rough set–theoretic ideas of approximation. This fact prompts us even more towards a discussion of synthesis of concepts pertaining to Spatial Reasoning by means of Rough Mereology.

We propose to begin in Section 2 with a brief introduction to Rough Set Theory. Then we propose to introduce in respective Sections 3 and 4 Ontology, respectively, Mereology along with interpretations of these theories in Information Systems. Rough Mereology is discussed in Section 5 with examples of rough inclusions induced in information systems. Section 6 introduces basic aspects of Qualitative Spatial Reasoning and in Section 7 devoted to Mereotopology we show that rough inclusions induce quasi–Čech topologies which in certain models become naturally quasi–topologies (i.e. topologies without the null element). Section 8 outlines how one may define basic primitives of geometry by means of rough inclusions and this paper concludes with Section 9 with examples concerning rough mereo–topology as well as rough mereo–geometry.

2. Rough Set Theory

Rough Set Theory has to do with data represented as an *information system*; the information system is formally presented as a pair $\mathsf{A}=(U,A)$ where U is a universe of objects (represented as rows in the *information table* $U \times A$) and A is a set of *attributes* (represented as columns in the *information table* $U \times A$).

Each attribute $a \in A$ is formalized as a function $a : U \longrightarrow V_a$ where V_a is the *value set* of a. We assume that for any a the set V_a is finite; an attribute a is *essential* when V_a is at least a two–element set and in what follows all attributes are assumed to be essential.

The ontological assumption about the information system (U, A) is that it does represent the whole content of knowledge about the real world to which the data refer. A fortiori, any inference about the world is to be made on the basis of the knowledge represented by the given information system and what bears on the knowledge content is actually not individual objects (rows) but rather their information sets.

For an object $u \in U$, and a set $B \subseteq A$, we call an *information set* of u the set $Inf_B(u) = \{(a, a(u)) : a \in B\}$.

In consequence of the above assumption, we identify any two objects (rows) whose information sets are identical; formally, we define the *B–indiscernibility relation* IND_B as follows:

$$(u, u') \in IND_B \Longleftrightarrow Inf_B(u) = Inf_B(u').$$

Then, the B–indiscernibility classes $[u]_B$ are represented by information sets $Inf_B(u)$ in the one–one way.

Among concepts (i.e. sets of objects) one has to make a distinction: there are concepts which may be described in terms of attributes and their values completely and certainly and there are concepts whose description is by necessity uncertain and incomplete.

Now, given $B \subseteq A$ ($B \neq \emptyset$), we may define a *B–exact concept* as a concept X such that X may be represented as a union of B–indiscernibility classes i.e. **if** $u \in X \wedge (u, u') \in IND_B$ **then** $u' \in X$.

It may happen that, given $\mathsf{A} = (U, A)$, we are interested also in some of its extensions of the form $\mathsf{A}^\infty = (U^\infty, A^\infty)$ where $A \subseteq A^\infty$, $B \subseteq B^\infty$ and A^∞, U^∞ are infinite (or, potentially infinite). In such case, the following notion makes sense.

An *A–elementary exact concept* will be defined now as a B-exact concept for some B (we assume that such B is finite).

Returning to B, we may now describe all concepts approximately in terms of B–attributes and their values, viz. given a non–empty concept $X \subseteq U$, we define:

the B–lower approximation $B_- X = \{u \in U : [u]_B \subseteq X\}$;
the B–upper approximation $B^+ X = \{u \in U : [u]_B \cap X \neq \emptyset\}$.

For each concept X, one may now describe X approximately–by means of knowledge represented by the set B of attributes– as the pair $(B_- X, B^+ X)$ of two B–exact sets. The general properties of this approximation may be found in [61].

Clearly, a concept X is a B–exact concept if and only if the condition $B_- X = B^+ X$ holds. Otherwise, X is said to be a *B–in–exact concept.* More generally, we may extend this definition taking into account all classes $[u]_B$, any B : for a concept X, we let

the A–lower approximation $\mathsf{A}_- X = \{u \in U : \exists B.[u]_B \subseteq X\}$;
the A–upper approximation $\mathsf{A}^+ X = \{u \in U : \forall B.[u]_B \cap X \neq \emptyset\}$.

We will call a concept X an *A–exact concept* in case $\mathsf{A}_- X = \mathsf{A}^+ X$; otherwise, X will be called an *A–in–exact concept.* Clearly, any elementary A–exact concept is an A–exact concept.

One may give a logical frame to our discussion cf. [61], [76]. Let us call a *descriptor* any pair of the form (a, v) where $a \in A$ and $v \in V_a$; descriptors may be regarded as elementary formulae from which formulae of the *descriptor logic* are formed by means of the propositional connectives $\vee$, $\wedge$, $\neg$ (let us observe that negation is not necessary in the finite case). The meaning $[\alpha]$ of a formula α is defined inductively:

$[(a, v)] = \{u \in U : a(u) = v\}$;
$[\alpha \vee \beta] = [\alpha] \cup [\beta]$;
$[\alpha \wedge \beta] = [\alpha] \cap [\beta]$;
$[\neg \alpha] = U - [\alpha]$.

For any pair $B \subseteq A, v = Inf_B(u)$, where $u \in U$, the formula $\alpha_{B,v}$ is the conjunction $\wedge_{a \in B}(a, v_a)$; clearly, $[\alpha_{B,v}] = [u]_B$.

In the sequel, we will call a pair (B, v) a *template*. Templates (defined also equivalently as $\alpha_{B,v}$) play an important role in rough set–theoretic methods in Knowledge Discovery and Data Mining cf. a throughout discussion in [74], [73], [8].

In the above discussion, we actually have introduced several types of concepts (names) like an A–exact set, B–exact set, etc. and we have also some examples of objects (represented as subsets of U) which answer to some of these names. For instance, when $U = \{u_1, u_2\}$, $A = \{a_1, a_2\}$ and $V_a = \{0, 1\}$ for each a, then the set $X=\{u : a_1(u) = 0\}$ is both A–exact and B–exact where $B = \{a_1\}$. In an informal way, we would write these facts down using phrases of the form: "*X is B*–exact" etc.

In application-oriented spatial reasoning systems, ontology appears as typology of concepts and their successive taxonomy cf. e.g. [56] (to quote a small excerpt: *edge* is *frontier*, *barrier*, *dam*, *cliff*, *shoreline*).

The informal connective "is" may undergo a formalization in which it is given a precise meaning. So we now give a formal scheme of Ontology and next we show that it does agree with our taxonomy presented above. The ontological scheme we choose to apply here is the Leśniewski Ontology cf. [37], [49], [52], [79].

We first give an introduction to formal Ontology and then we propose some interpretations of this formal scheme in Information Systems.

3. Ontology

Ontology was intended by Stanisław Leśniewski as a formulation of general principles of *being* [52] cf. also [38], [49], [79], [37].

The only primitive notion of Ontology of Leśniewski is the copula "*is*" denoted by the symbol ε.

We will present now in a nutshell the scheme of Elementary Ontology. Formulas of Ontology will be constructed in the standard predicate calculus. We begin with the axiom of Ontology i.e. a formula which introduces the copula ε.

3.1. Axiom of Ontology

The original Axiom of Ontology defining the meaning of ε is as follows

The ontological axiom

$$X \varepsilon Y \Longleftrightarrow \exists Z. Z \varepsilon X \wedge \forall U, W.(U \varepsilon X \wedge W \varepsilon X \Longrightarrow U \varepsilon W) \wedge \forall Z.(Z \varepsilon X \Longrightarrow Z \varepsilon Y).$$

In this axiom, the defined copula ε happens to occur in both sides of the equivalence: however, the definiendum $X \varepsilon Y$ belongs in the left side only and we may perceive the axiom as the definition of the meaning of $X \varepsilon Y$ via the meaning of terms of "lower level" $Z \varepsilon X$, $Z \varepsilon Y$ etc.

According to this reading of the axiom, the proposition $X\varepsilon Y$ is true if and only if the conjunction holds of the following three propositions:

1. $\exists Z.Z\varepsilon X$

   this proposition asserts the existence of an object (name) Z which is X and so X is not an empty name;
2. $\forall U,W.(U\varepsilon X \wedge W\varepsilon X \Longrightarrow U\varepsilon W)$

   this proposition asserts that any two objects which are X are each other: this means that X is an individual name (representable as a singleton);
3. $\forall Z.(Z\varepsilon X \Longrightarrow Z\varepsilon Y)$

   this proposition asserts that every object which is X is Y as well (or, X is *contained* in Y).

The meaning of $X\varepsilon Y$ can be made clear now: X is an individual (name) and this individual is Y (i.e. belongs in Y, responds to the name of Y, etc.). In particular, the formula $X\varepsilon X$ denotes that X is an individual.

We introduce the identity of individuals via

$$X\varepsilon = (Y) \Longleftrightarrow X\varepsilon X \wedge Y\varepsilon Y \wedge X\varepsilon Y \wedge Y\varepsilon X.$$

3.2. Ontology in Information Systems

We propose to discuss here two types of template–based Ontology, related to each other.

3.2.1. Ontology of B–indiscernibility

We form first individual entities. Recall that the symbol $[u]_B$ denotes the indiscernibility class of u over B. We would like to propose an ontology based on the notion of a template in which individual objects would be B–exact concepts. We would like to introduce B-*names* as distributive classes (sets) of B–individuals. Then, $X\varepsilon Y$ means that X is a B-individual and Y is a name for a set of such individuals containing X.

We would like to make use of templates and given $C \subseteq B$ and $v = Inf_C(u)$ for some $u \in U$, we will interpret the template (C, v) in a two–fold way.

1. We may take the meaning $[C, v]$ of the template (C, v) as the set $\{u \in U : \forall a \in C.a(u) = v_a\}$;
2. We may take (C, v) as the name for the set of all $[D, w]$ such that $C \subseteq D \wedge w|C = v$ and $[D, w] \neq \emptyset$. This defines the meaning of the formula

$$[D, w]\varepsilon(C, v).$$

In this way we could create names for classes (sets) without the need of specifying these sets. We may propose a functor + defined as:

$$X\varepsilon(C,v)+(D,w) \iff X\varepsilon(C,v) \vee X\varepsilon(D,w)$$

and then we may write down any name as $+_{i=1}^{k}(C_i,v_i)$.

Now, we will define our objects of interest; we will render the complex interplay between the structure and name of an object by defining our objects as pairs (*concept structure*, *name*). Thus our objects will be pairs of the form

$$(<[D_i,w_i]>_{i=1}^{m}, +_{j=1}^{k}(C_j,v_j))$$

subject to the following requirements (structure is represented by the sequence of constituents we prefer to single out)

1. $\forall i \in \{1,...,m\}.\exists j \in \{1,...,k\}.[D_i,w_i]\varepsilon(C_j,v_j)$;
2. $\forall j \in \{1,...,k\}.\exists i \in \{1,...,m\}.[D_i,w_i]\varepsilon(C_j,v_j)$.

We avoid, as witnessed by 2., creating excessively complex names.

Let us observe that: $[D,w]\varepsilon(B,v) \implies [D,w]=[B,v]$ and in consequence of this we will identify $[B,v]$ and (B,v). Hence $[B,v]\varepsilon[B,v]$ implying that $[B,v]$ is an *individual* object.

We now define the meaning of ε for more complex objects. To this end, for objects

$\mathcal{A} : (<[D_i,w_i]>_{i=1}^{m}, +_{j=1}^{k}(C_j,v_j))$ and $\mathcal{B} : (<[E_u,t_u]>_{u=1}^{p}, +_{q=1}^{r}(F_q,v_q))$ we let:

$$\mathcal{A}\varepsilon\mathcal{B} \iff$$

1. $\forall i \in \{1,...,m\}.\exists q \in \{1,...,r\}.[D_i,w_i]\varepsilon(F_q,v_q)$;
2. $\forall q \in \{1,...,r\}.\exists i \in \{1,...,m\}.[D_i,w_i]\varepsilon(F_q,v_q)$.

Again, we have:

$$(<[D_i,w_i]>_{i=1}^{m}, +_{j=1}^{k}(C_j,v_j))\varepsilon(<[B,t_q]>_{q=1}^{r}, +_{q=1}^{r}(B,t_q)) \implies$$

$$<[D_i,w_i]>_{i=1}^{m}=<[B,t_q]>_{q=1}^{r} \wedge +_{j=1}^{k}(C_j,v_j) = +_{q=1}^{r}(B,t_q)$$

implying that $(<[B,t_q]>_{q=1}^{r}, +_{q=1}^{r}(B,t_q))$ is an *individual*.

In this way, templates may be used to define a formal ontology of concepts in which B–exact concepts represented canonically as unions of B–indiscernibility classes are singled out as individuals; in Section 4.2, for these objects the notion of a part will be defined.

3.2.2. Ontology of A–indiscernibility

This case is distinct from the previous one when the attribute set A is an infinite subset of A^{∞} and the value set V_a has cardinality ≥ 2 for infinitely many $a \in A$. Adopting the ontology from the previous section, we may see that in this case there are no individual objects: we obtain an infinite forest of names.

4. The Leśniewski Mereology

Mereology is a theory of collective classes i.e. individual entities representing general names as opposed to Ontology which is a theory of distributive classes i.e. general names. The distinction between a distributive class and its collective class counterpart is like the distinction between a family of sets and its union being a set. For instance, we may represent United States through the list of its states (i.e. as a distributive class) or we may mean "United States" as the collective class of its states e.g. their union (i.e. as the individual having the spatial location in the northern hemisphere and containing as parts all of its states).

This explains the purpose of Mereology which is to express individual objects though their parts and to define some complex objects as individuals by means of parts of their individual components.

Mereology may be based on each of a few notions like those of a *part*, an *element (ingredient)*, a *class* etc. Historically it has been conceived by Stanisław Leśniewski [51], [53], [54] cf. [20], [89], [83], [84], as a theory of the relation *part* and we here follow this line of development. In particular, we present the development of Mereology within Ontology: names of mereological constructs will be formed by means of ontological rules. The meaning of the copula ε explained above, we may now make use of this notation in what follows.

We assume that the copula ε is given and that the Ontology Axiom holds. Under these assumptions, we introduce the notion of a predicate *pt* of *part*. Our presentation is based on [53] in the first place.

4.1. Mereology axioms

(A1) $X \varepsilon pt(Y) \Longrightarrow X \varepsilon X \wedge Y \varepsilon Y$

this means that the predicate pt is defined for `individual entities` only;

(A2) $X \varepsilon pt(Y) \wedge Y \varepsilon pt(Z) \Longrightarrow X \varepsilon pt(Z)$

meaning that the predicate pt is transitive;

(A3) $\neg(X \varepsilon pt(X))$

which means that the predicate pt is non–reflexive.

On the basis of the notion of part, we define the notion of an *element* (an improper, possibly, part; called originally, an *ingredient*) as a predicate el : $X \varepsilon el(Y) \Longleftrightarrow X \varepsilon pt(Y) \vee X = Y$.

The remaining axioms of mereology are related to the class functor which converts distributive classes (general names) into individual entities. The class operator Kl is a principal tool in applications of Rough Mereology to problems of Distributed Systems, Knowledge Granulation, Computing with Words where it does play the role of granulating (clustering) operator allowing for forming granules of knowledge and subsequently instrumental in calculi on them cf. [64], [65], [66], [67], [75].

We may now introduce the notion of a (collective) class via the class functor Kl.

The class operator

$$X \varepsilon Kl(Y) \iff$$
$$\exists Z.Z\varepsilon Y \wedge \forall Z.(Z\varepsilon Y \implies Z\varepsilon el(X)) \wedge \forall Z.(Z\varepsilon el(X) \implies \exists U, W.(U\varepsilon Y \wedge W\varepsilon el(U) \wedge W\varepsilon el(Z)).$$

Let us disentangle the meaning of this definition. First, we may realize that the class operator Kl is intended as the operator converting names (general sets of entities) into individual entities i.e. collective classes; its role may be fully compared to the role of the union of sets operator in the classical set theory. The analogy is indeed not only functional but also formal.

Let us look at the subsequent conjuncts in the defining formula above.

1. $\exists Z.Z\varepsilon Y$

   this means that Y is a non–empty name (recall that the union of the empty family of sets is the empty set hence prohibited in Ontology);
2. $\forall Z.(Z\varepsilon Y \implies Z\varepsilon el(X))$

   meaning that any individual listed in Y is an element of $Kl(Y)$ (compare with: any element of the family of sets is a subset of the union of that family);
3. $\forall Z.(Z\varepsilon el(X) \implies \exists U, W.(U\varepsilon Y \wedge W\varepsilon el(U) \wedge W\varepsilon el(Z))$

   this means that any element of $Kl(Y)$ has an element in common with an individual in Y (similarly, any element in the union of a family of sets is an element in at least one member of this family).

Thus, the class functor pastes together individuals in Y by means of their common elements. The class functor is subject to the following postulates.

(A4) $X\varepsilon Kl(Y) \wedge Z\varepsilon Kl(Y) \implies X\varepsilon Z$

this means that $Kl(Y)$ is an individual name (entity), for any (non–empty) Y;

(A5) $\exists Z.Z\varepsilon Y \iff \exists Z.Z\varepsilon Kl(Y)$

meaning that $Kl(Y)$ exists (i.e. is a non–empty individual name) if and only if Y is a non–empty name.

One may prove cf. [53] the following inference rule.

Proposition 4.1.

$$[\forall Z.(Z\varepsilon el(X) \longrightarrow \exists T.(T\varepsilon el(Z) \wedge T\varepsilon el(Y))] \longrightarrow X\varepsilon el(Y).$$

4.2. Mereology in Information Systems

As with Ontology (cf. 3.2), we discern between two basic types of mereological structures. First, we discuss the *template mereology*.

4.2.1. Mereology of B–indiscernibility

Recall (cf. Section 3.2) that individual entities in this case are B–exact sets and general names are collections (sets, lists) of B–exact sets.

We now define the predicate pt on individual objects: for individuals

$$\mathcal{A} : (< [B, v_i] >_{i=1}^{m}, +_{i=1}^{m}(B, v_i)) \ and \ \mathcal{B} : (< [B, w_j] >_{j=1}^{k}, +_{j=1}^{k}(B, w_j))$$

we let $\mathcal{A}\varepsilon pt(\mathcal{B})$ if and only if

1. $\forall i.\exists j.[B, v_i] = [B, w_j]$;
2. $\exists Z \varepsilon B.\neg(Z \varepsilon A)$ for some individual Z.

We may now check easily via the definition of a class (cf. page 9) that the class of a name $Y : (< [D_i, w_i] >_{i=1}^{m}, +_{j=1}^{k}(C_j, v_j))$ i.e. the individual $Kl(Y)$ is the individual $X : (< [B, w_q] >_{q=1}^{r}, +_{q=1}^{r}(B, v_q))$ where $[B, w_q]$ runs over all indiscernibility classes $[B, w]$ such that $\exists j.[B, w]\varepsilon(C_j, v_j)$. Thus, $Kl(Y)$ may be identified with the B-exact concept $\bigcup_{q=1}^{r}[B, w_q]$.

The above example illustrates the mechanism by which the class operator works in perspective of its applications in Section 7 and in Section 8.

4.2.2. Mereology of A–indiscernibility

In the context of Section 3.2, we do not have individuals hence no part predicate may be defined. In what follows, we restrict ourselves to finite information systems.

5. Rough Mereology

Rough Mereology is an extension of Mereology based on the predicate of being a part in a degree; this predicate is rendered here as a family μ_r parameterized by a real parameter $r \in [0, 1]$ with the intent that $X\varepsilon\mu_r(Y)$ reads "*X is a part of Y in degree r*". We begin with the set of axioms and we construct the axiom system as an extension of systems for Ontology and Mereology 4.

We assume thus that a predicate *el* of an element satisfying the Mereology axiom system is given; around this, we develop a system of axioms for Rough Mereology.

5.1. The axiom system

The following is the list of basic postulates.

(RM1) $X\varepsilon\mu_1(Y) \Longleftrightarrow X\varepsilon el(Y)$

this means that being a part in degree 1 is equivalent to being an element: this establishes the connection between Rough Mereology and Mereology;

(RM2) $X\varepsilon\mu_1(Y) \Longrightarrow \forall Z.(Z\varepsilon\mu_r(X) \Longrightarrow Z\varepsilon\mu_r(Y))$

meaning the monotonicity property: any object Z is a part of Y in degree not smaller than that of being a part in X whenever X is an element of Y;

(RM3) $X = Y \wedge X \varepsilon \mu_r(Z) \Longrightarrow Y \varepsilon \mu_r(Z))$
this means that the identity of individuals is a congruence with respect to μ;
(RM4) $X \varepsilon \mu_r(Y) \wedge s \leq r \Longrightarrow X \varepsilon \mu_s(Y)$
establishes the meaning " a part in degree at least r".

It follows that the predicate μ_1 coincides with the given predicate *el* establishing a link between Rough Mereology and Mereology while predicates μ_r with $r < 1$ diffuse *el* to a graded family of predicates expressing being an element (or, part) in various degrees.

5.2. Rough Mereology in Information Systems

We will refer to Section 4.2, where examples of mereological decompositions into parts were given. We will propose here a few measures of rough partial containment based on either individual frequency count or on attribute–value frequency count. For simplicity sake, we will denote inividual objects by their concept coordinate only i.e. as $\bigcup_{i=1}^{k} [B, v_i]$, or, equivalently as $Kl(+_{i=1}^{k}(B, v_i)$.

5.2.1. Rough membership functions

Here, we apply the idea of a rough membership function of Pawlak and Skowron [63]; we recall that given a subset (in the set–theoretic sense) X of the universe U of an information system A=(U, A), one defines the rough membership function μ_X^{A} by letting

$$\mu_X^{\mathsf{A}}(u) = \frac{card(X \cap [u]_A)}{card([u]_A)} \quad \text{for} \quad u \in U.$$

This notion may be extended [64] to a notion of a rough membership function defined on concepts (i.e. subsets of U): given two non–empty concepts X, Y, we let:

$$\mu_X(Y) = \frac{card(X \cap Y)}{card(Y)}.$$

Thus, $\mu_X(Y)$ is a measure of the degree in which Y is contained in X.

5.2.2. Rough Mereology of B–exact sets: the row frequency count

We apply the rough membership function in its generalized form. Given two B–exact sets, X, Y, we let $r = \mu_X(Y)$ and accordingly $Y \varepsilon \mu_r(X)$. This measure μ is thus based on the frequency count of rows of the information table in respectively, X and Y. From probabilistic point of view, it may be regarded as an unbiased estimate of the conditional probability $Pr(X|Y)$ cf. [62].

5.2.3. Rough Mereology of B–exact sets: the B–class frequency count

Here, we propose yet another measure based on the rough membership function; we apply it counting this time the number of B–indiscernibility classes in $X \cap Y$ and Y, respectively; for $X = Kl(+_{i=1}^{k}(B, v_i))$ and $Y = Kl(+_{j=1}^{m}(B, w_j))$, Section 4.2, we let:

$$r = \frac{card(\{[B, v_i] : i \leq k\} \cap \{[B, w_j] : j \leq m\})}{m} \quad \text{and accordingly,} \quad Y \varepsilon \mu_r(X).$$

5.2.4. Rough Mereology of A–exact sets

We may apply here either of the two frequency count measures defined earlier: in the first case, for two A–exact sets $X = Kl(+_{i=1}^{k}[B_i, v_i])$ and $Y = Kl(+_{j=1}^{m}[C_j, w_j])$, we let:

$$r = \frac{card(X \cap Y)}{card(Y)} \text{ and accordingly, } Y \varepsilon \mu_r(X)$$

where cardinalities are counted over the subsystem $(U, \bigcup_i B_i \cup \bigcup_j C_j)$.

In the second case, we let:

$$r = \frac{card(\{[B_i, v_i] : i \leq k\} \cap \{[C_j, w_j] : j \leq m\})}{m} \text{ and accordingly, } Y \varepsilon \mu_r(X).$$

We now propose a method for extending a measure defined for elements of two concepts to a measure on these two concepts; this idea will be applied in the following section.

5.2.5. Extending rough inclusions

Assume that we are given two individuals X, Y being classes of (finite) names: $X = Kl(X')$, $Y = Kl(Y')$ and that we have defined values of μ for pairs T, Z of individuals where $T \varepsilon X'$, $Z \varepsilon Y'$.

We extend μ to a measure μ^* on X, Y by letting:

$r = min_{Z \in Y'}\{max_{T \in X'} max\{s : Z \varepsilon \mu_s(T)\}\}$ and $Y \varepsilon \mu^*_r(X)$.

It may be proved straightforwardly that

Proposition 5.1. *The measure μ^* satisfies (RM1)–(RM4).*

We now have at our disposal some recipes for introducing rough inclusions in information systems. The choice may depend on the context; let us observe that we may also have some parameterized formulae, subject to optimization in a given context.

5.3. Renormalization by necessity

We introduce now, following Polkowski and Skowron [64], a modification to our functors μ_r; it is based on an application of residuated implication [41] and a measure of containment defined within the fuzzy set theory (the necessity measure) [36], [7]. Combining the two ideas, we achieve a formula for μ_r which allows for a transitivity rule; this rule will in turn allow to introduce into our universe rough mereological topologies.

We therefore recall the notion of a *t–norm* $\top$ as a function of two arguments $\top : [0,1]^2 \longrightarrow [0,1]$ which satisfies the following requirements:

1. $\top(x, y) = \top(y, x)$;
2. $\top(x, \top(y, z)) = \top(\top(x, y), z)$;
3. $\top(x, 1) = x$;
4. $x' \geq x \wedge y' \geq y \Longrightarrow \top(x', y') \geq \top(x, y)$.

We also invoke a notion of fuzzy containment $\subset_r$ based on *necessity* cf. [36]; it relies on a many-valued implication Υ i.e. on a function $\Upsilon : [0,1]^2 \longrightarrow [0,1]$ according to the formula:

$$X \subset_r Y \Longleftrightarrow \forall Z.(\Upsilon(\mu_X(Z), \mu_Y(Z)) \geq r)$$

where μ_A is the fuzzy membership function [41] of the fuzzy set A.

We replace Υ with a specific implication viz. the residuated implication $\overrightarrow{\top}$ induced by $\top$ and defined by the formula: $\overrightarrow{\top}(r,s) \geq t \Longleftrightarrow \top(t,r) \leq s$.

We define a predicate $\mu_{\top,r}$ where $r \in [0,1]$, according to the formula:

$$X \varepsilon \mu_{\top,r}(Y) \Longleftrightarrow X \varepsilon X \wedge \forall Z.(\exists t, w. Z \varepsilon \mu_t(X) \wedge Z \varepsilon \mu_w(Y) \wedge \overrightarrow{\top}(t,w) \geq r.)$$

It turns out, as first proved in a different context in [64], that $\mu_{\top,r}$ satisfies axioms (RM1-RM4); we include the proof in our case.

Proposition 5.2. *Predicates $\mu_{\top,r}$ satisfy (RM1)-(RM4)*

Proof:
For (RM1): assume that $X \varepsilon \mu_{\top,1}(Y)$ so for any Z from $Z \varepsilon \mu_u(X), Z \varepsilon \mu_v(Y)$ it follows that $\overrightarrow{\top}(u,v) \geq 1$ hence $\top(1,u) \leq v$ i.e. $u \leq v$; this implies that $\forall Z.(Z \varepsilon \mu_1(X) \Longrightarrow Z \varepsilon \mu_1(Y))$ i.e. $\forall Z.(Z \varepsilon el(X) \Longrightarrow Z \varepsilon el(Y))$ and thus $X \varepsilon el(Y)$. Conversely, $X \varepsilon el(Y)$ implies $\forall Z.(Z \varepsilon \mu_u(X) \wedge Z \varepsilon \mu_v(Y) \Longrightarrow u \leq v$ so $\overrightarrow{\top}(u,v) \geq 1)$ and finally $X \varepsilon \mu_{\top,1}(Y)$.

For (RM2), let $X \varepsilon \mu_{\top,1}(Y)$ and $Z \varepsilon \mu_{\top,u}(X)$; for any T from $T \varepsilon \mu_\alpha(Z)$, $T \varepsilon \mu_v(X)$, $T \varepsilon \mu_w(Y)$ it follows that $v \leq w$ hence $\overrightarrow{\top}(\alpha,w) \geq \overrightarrow{\top}(\alpha,v)$ and $\overrightarrow{\top}(\alpha,v) \geq u$ implies $\overrightarrow{\top}(\alpha,w) \geq u$ i.e. $Z \varepsilon \mu_{\top,u}(Y)$.

In case of (RM3), assume that $X = Y$; then for any T, $\alpha : T \varepsilon \mu_\alpha(X) \Longleftrightarrow T \varepsilon \mu_\alpha(Y)$ hence for any T from $T \varepsilon \mu_\gamma(X), T \varepsilon \mu_\delta(Z)$ with $\overrightarrow{\top}(\gamma,\delta) \geq r$ it follows that $T \varepsilon \mu_\gamma(Y), T \varepsilon \mu_\delta(Z)$ with $\overrightarrow{\top}(\gamma,\delta) \geq r$ i.e. $X \varepsilon \mu_{\top,r}(T) \Longrightarrow Y \varepsilon \mu_{\top,r}(T)$.

(RM4) is obviously satisfied by virtue of definition of $\mu_{\top,r}$. □

An advantage of this rough inclusion is the fact that it does satisfy a deduction rule of the form

$$(DR)\frac{X \varepsilon \mu_r(Y), Y \varepsilon \mu_s(Z)}{X \varepsilon \mu_u(Z)}$$

where $u = f(r,s)$ depends functionally on r, s. Clearly, the obvious candidate for f is $\top$. Again, we include a short proof for completeness sake.

Proposition 5.3. *(DR) holds in the form :*

$$\frac{X \varepsilon \mu_{\top,r}(Y), Y \varepsilon \mu_{\top,s}(Z)}{X \varepsilon \mu_{\top,\top(r,s)}(Z)}.$$

Proof:
Assume that $X\varepsilon\mu_{\top,r}(Y)$, $Y\varepsilon\mu_{\top,s}(Z)$; then: $T\varepsilon\mu_{\alpha}(X)$, $T\varepsilon\mu_{\beta}(Y)$, $T\varepsilon\mu_{\delta}(Z)$ imply $\overrightarrow{\top}(\alpha,\beta) \geq r$, $\overrightarrow{\top}(\beta,\delta) \geq s$ i.e. $\top(r,\alpha) \leq \beta, \top(s,\beta) \leq \delta$ and monotonicity of $\top$ implies $\top(s,\top(r,\alpha)) \leq \delta$ so $\top(\top(s,r),\alpha)) \leq \delta$ and $\top(\top(r,s),\alpha)) \leq \delta$ which finally yields $\overrightarrow{\top}(\alpha,\delta) \geq \top(r,s)$ so $X\varepsilon\mu_{\top,\top(r,s)}(Z)$. □

We now propose to synthesize basic topological and geometric constructs applied in Qualitative Spatial Reasoning based currently on Connection approach in Mereology e.g. [6], [5] by means of Rough Mereology.

6. Introduction to Qualitative Spatial Reasoning

Qualitative Reasoning aims at studying concepts and calculi on them that arise often at early stages of problem analysis when one is refraining from qualitative or metric details cf. [16]; as such it has close relations to the design cf. [11] as well as planning stages cf. [31] of the model synthesis process. Classical formal approaches to spatial reasoning i.e. to representing spatial entities (points, surfaces, solids etc.) and their features (dimensionality, shape, connectedness degree etc.) rely on Geometry or Topology i.e. on formal theories whose models are spaces (universes) constructed as sets of points; contrary to this approach, qualitative reasoning about space often exploits pieces of space (regions, boundaries, walls, membranes etc.) and argues in terms of relations abstracted from a commonsense perception (like *connected*, *discrete from*, *adjacent*, *intersecting*). In this approach, points appear as ideal objects (e.g. ultrafilters of regions/solids [88]). Qualitative Spatial Reasoning has a wide variety of applications, among them, to mention only a few, representation of knowledge, cognitive maps and navigation tasks in robotics (e.g. [42], [43], [44], [1], [3], [23], [39], [28]), Geographical Information Systems and spatial databases including *Naive Geography* (e.g. [26], [27], [35], [24]), high-level Computer Vision (e.g. [94]), studies in semantics of orientational lexemes and in semantics of movement (e.g. [6], [5]). Spatial Reasoning establishes a link between Computer Science and Cognitive Sciences (e.g. [29]) and it has close and deep relationships with philosophical and logical theories of space and time (e.g. [70], [9], [2]). A more complete perspective on Spatial Reasoning and its variety of themes and techniques may be acquired by visiting one of the following sites: [85], [93], [58].

Any formal approach to Spatial Reasoning, however, would require a formal approach to Ontology as well cf. [34], [80], [12]. In this Chapter we adopt as formal Ontology the ontological theory of Stanisław Leśniewski (cf. [51], [52], [79], [49], [38], [20]. This theory is briefly introduced in Section 3.

For expressing relations among entities, mathematics proposes two basic languages: the language of set theory, based on the opposition element–set, where distributive classes of entities are considered as sets consisting of (discrete) atomic entities, and languages of mereology, for discussing entities continuous in their nature, based on the opposition part–whole. It is thus not surprising that Spatial Reasoning relies to great extent on mereological theories of part cf. [4], [5], [6], [14], [17], [32], [33], [30], [81], [82], [57].

Mereological ideas have been early applied toward axiomatization of geometry of solids cf. [47], [88]. Mereological theories dominant nowadays come from ideas proposed independently by Stanisław Leśniewski and Alfred North Whitehead.

Mereological theory of Leśniewski is based on the notion of a part (proper) and the notion of a (collective) class cf. [51], [53], [20], [83], [54]. Mereological ideas of Whitehead based on the dual to part notion of an extension [95] were formulated in the Calculus of Individuals [48] and given a formulation in terms of the notion of a Connection [14]. Mereology based on connection gave rise to spatial calculi based on topological notions derived therefrom (mereotopology) cf. [18], [16], [22], [25], [5], [6], [17], [32], [33], [30], [81], [57].

In this paper, we demonstrate that in the framework of Rough Mereology one may define a quasi–Čech topology [21] (a quasi– topology was introduced in the connection model of Mereology [14], [5] under additional assumptions of regularity). By a *quasi–topology* we mean a topology without the null element (being the equivalent of the empty set).

Finally, we apply Rough Mereology toward inducing geometrical notions. It is well known cf. e.g. [89], [9] that geometry may be introduced via notions of nearness, betweenness etc. In Section 8, we define these notions by means of a rough mereological notion of distance and we show that in this way a geometry may be defined in the rough mereological universe. This geometry is clearly of approximate character, approaching precise notions in a degree due to uncertainty of knowledge encoded in rough inclusions.

As rough mereological constructs (rough inclusions) may be induced from data tables (information systems), as indicated in Section 5, reasoning about spatial entities by means of rough mereology may be carried out on the basis of data (e.g. in spatial databases).

7. Mereotopology

We now are concerned with topological structures arising in mereological universe endowed with a rough inclusion.

Topological structures may be defined within the connection framework via the notion of a *non–tangential part* and interior entities are formed then by means of some fusion operators cf. e.g. [5], [57]. The functor of connection allows also for some calculi of topological character based directly on regions e.g. $RCC-calculus$ cf. [33]. For a different approach where connection may be derived from the axiomatized notion of a boundary cf. [82].

These topological structures provide an environment in which it is possible to carry out spatial reasoning.

7.1. Mereotopology : Čech topologies

It has been shown that in mereological setting a quasi–Čech topology may be defined (cf. [14]) which under additional assumptions (op.cit.) may be made into a quasi–topology. Here, we induce a quasi–Čech topology (i.e. topology without the null object) in any rough mereological universe.

The Čech topology [21], [46] is a weak topological structure as it is required here only that the closure operator satisfy the following:

1. (ČCl1) $cl\emptyset = \emptyset$;
2. (ČCl2) $X \subseteq clX$;
3. (ČCl3) $X \subseteq Y \Longrightarrow clX \subseteq clY$.

so the associated Čech interior operator int should only satisfy the following: $int\emptyset = \emptyset$; $intX \subseteq X$; $X \subseteq Y \Longrightarrow intX \subseteq intY$.

Čech topologies arise naturally in problems related to information systems when one considers coverings induced by similarity relations instead of partitions induced by indiscernibility relations [55].

We now introduce a quasi–Čech topology into a rough mereological universe: we may remember that in our context of Ontology, the empty set (name) may not be used.

To this end, we define the class Kl_rX for any object X and $r < 1$ by the following. First, we introduce a name (M_rX) for the property of being a part in a degree r:

$$Z\varepsilon M_rX \Longleftrightarrow Z\varepsilon Z \wedge Z\varepsilon\mu_r(X).$$

Now, we define the individual entity Kl_rX:

$$Z\varepsilon Kl_rX \Longleftrightarrow Z\varepsilon Z \wedge Z\varepsilon M_rX.$$

Thus Kl_rX is the class of objects having the property $\mu_r(X)$. We now give a direct characterization of Kl_rX. With this aim, we introduce a name B_rX defined by means of the condition:

$$Z\varepsilon el(B_rX) \Longleftrightarrow \exists T(Z\varepsilon el(T) \wedge T\varepsilon\mu_r(X)).$$

This definition is correct, as $B_rX = Kl(el(B_rX))$. Then we have

Proposition 7.1. $Kl_rX = B_rX$

Proof:
Assume first that $Z\varepsilon el(Kl_rX)$; then by the class definition, for some U, W, we have $U\varepsilon el(Z)$, $U\varepsilon el(W)$, $W\varepsilon\mu_r(X)$, hence $U\varepsilon el(B_rX)$ and the inference rule (4.1) implies that $Kl_rX\varepsilon el(B_rX)$.

Conversely, $Z\varepsilon el(B_rX)$ implies that for some T we have $Z\varepsilon el(T)$, $T\varepsilon\mu_r(X)$ hence by he class definition $T\varepsilon el(Kl_rX)$ and so $Z\varepsilon el(Kl_rX)$ implying by the inference rule that $B_rX\ \varepsilon\ el(Kl_rX)$. Hence $Kl_rX = B_rX$. □

From this the following corollary follows

Corollary 7.1. *For $s \leq r$, $Kl_rX\varepsilon el(Kl_sX)$.*

Indeed, $Z\varepsilon el(Kl_rX)$ means that $Z\varepsilon el(T)$, $T\varepsilon\mu_r(X)$ for some T so by (RM4) we have $T\varepsilon\mu_s(X)$ and thus $Z\varepsilon el(Kl_sX)$. The corollary follows.

We mention yet a monotonicity property.

Proposition 7.2. $X\varepsilon el(Y) \Longrightarrow Kl_rX\varepsilon el(Kl_rY)$.

We admit B defined as follows as a base for open sets with which we define the interior operator:

$$Z\varepsilon B \Longleftrightarrow Z\varepsilon Z \wedge \exists X, r < 1. Z\varepsilon Kl_rX.$$

Following this we define a new predicate *int*. Again, we introduce first a new name $I(X)$:

$$Z\varepsilon I(X) \Longleftrightarrow Z\varepsilon Z \wedge \exists s < 1(Kl_sZ\varepsilon el(X)).$$

Now, we define the interior predicate *int*:

$$int(X) = Kl(I(X)).$$

Then we have the following properties of *int*.

Proposition 7.3. *For any* X, Y :

1. $int(X)\varepsilon el(X)$;
2. $X\varepsilon el(Y) \Longrightarrow int(X)\varepsilon el(int(Y))$;

Proof:
For (1): assume that $Z\varepsilon el(int(X))$; there exist U, W with $U\varepsilon el(Z)$, $U\varepsilon\ el(W)$, $Kl_sW\varepsilon el(X)$ for some $s < 1$; hence, $W\varepsilon el(X)$ and $U\varepsilon el(X)$ so the inference rule (4.1) implies that $int(X)\varepsilon el(X)$.

In case (2), assume that $X\varepsilon el(Y)$ and let $Z\varepsilon el(int(X))$. We have U, W with $U\varepsilon el(Z)$, $U\varepsilon el(W)$, $Kl_sW\varepsilon el(X)$ for an $s < 1$ hence $Kl_sW\varepsilon el(Y)$ and thus $W\varepsilon I(Y)$ hence $W\ \varepsilon\ el(int(Y))$ so a fortiori $U\varepsilon el(int(Y))$ so the inference rule implies that $int(X)\varepsilon el(int(Y))$.

□

Properties (1)-(2) witness that the quasi–topology introduced by B is a *quasi–Čech topology*. We denote it by the symbol τ_μ.

Proposition 7.4. *Rough mereotopology* τ_μ *induced by the rough inclusion* μ_r *is a quasi–Čech topology.*

We now study τ_μ under predicates $\mu_{\top,r}$; in this case, the quasi– Čech topology τ_μ turns out to be a quasi–topology.

7.2. Mereotopology: the case of $\mu_\top$

We begin with an application of deduction rule (DR) 5.3. We denote by the symbol $Kl_{\top,r}X$ the set $Kl_r X$ in case of the rough inclusion $\mu_\top$. We assume that $\top(r,s) < 1$ when $rs < 1$. We propose a new direct characterization of $Kl_{\top,r}X$.

Proposition 7.5. $Z\varepsilon el(Kl_{\top,r}X) \Longleftrightarrow Z\varepsilon\mu_{\top,r}(X)$.

Proof:
As $Z\varepsilon el(Kl_{\top,r}X)$ means that $Z\varepsilon\ el(T)$, $T\varepsilon\mu_{\top,r}(X)$ for some T hence $Z\varepsilon\mu_{\top,1}(T)$, $T\varepsilon\mu_{\top,r}(X)$ imply by (DR) that $Z\varepsilon\mu_{\top,\top(1,r)}(X)$ i.e. $Z\varepsilon\mu_{\top,r}(X)$. □

This Proposition means that $Kl_{\top,r}X$ may be regarded as "*an open ball of radius r centered at X*".

We assume now, additionally, that the t-norm $\top$ has the property that: for every $r < 1$ there exists $s < 1$ such that $\top(r,s) \geq r$. With this assumption, we have the following.

Proposition 7.6. *For* $Z\varepsilon el(Kl_{\top,r}(X))$,
if $s_0 = \arg\_\min\{s : \top(r,s) \geq r\}$ *then* $Kl_{\top,s_0}(Z)\varepsilon el(Kl_{\top,r}(X))$.

Proof:
Let $s \geq s_0$; consider $T\varepsilon el(Kl_{\top,s}(Z))$ so $T\varepsilon\mu_{\top,s}(Z)$. Then $T\varepsilon\mu_{\top,\top(s,r)}(X)$ hence $T\varepsilon\mu_{\top,r}(X)$ so $T\varepsilon el(Kl_{\top,r}(X))$ implying finally by the inference rule (4.1) that $Kl_{\top,s_0}(Z)\varepsilon\ el(Kl_{\top,r}(X))$. □

We define a predicate Ov (of rough mereological overlap) and a boolean predicate AND:

1. $Ov(X,Y) \Longleftrightarrow \exists Z.Z\varepsilon el(X) \wedge Z\varepsilon el(Y)$;
2. $Z\varepsilon el(AND(X,Y)) \Longleftrightarrow Ov(X,Y) \wedge Z\varepsilon el(X) \wedge Z\varepsilon el(Y)$.

Proposition 7.7. *The rough mereotopology* $\tau_{\mu_\top}$ *has the property:*
$AND(int(X), int(Y)) = int(AND(X,Y))$ *holds whenever*
$AND(int(X), int(Y))$ *is non-empty.*

Proof:
The intersection of two open basic classes may be described effectively 7.6: assume that $Z\varepsilon\ el(Kl_{\top,r}(X))$ and $Z\varepsilon\ el(Kl_{\top,s}(Y))$ for some $r,s < 1$. Then for $t_0 = \arg\_\min\{t : \top(r,t) \geq r, \top(s,t) \geq s\}$ and $1 > t \geq t_0$, we have $Kl_{\top,t}(Z)\varepsilon el(Kl_{\top,r}(X))$ and $Kl_{\top,t}(Z)\varepsilon el(Kl_{\top,s}(Y))$. The general case follows easily. □

Finally, we check that under our assumptions, the operator of interior is idempotent: $intint = int$ which will conclude our verification that the rough mereological topology is a topology.

Proposition 7.8. $int(int(X)) = int(X)$.

Proof:
It suffices to show that $int(X)\varepsilon el(int(int(X)))$ so we consider Z with $Z\varepsilon el(int(X))$. For some U,W, we have $U\varepsilon el(Z)$, $U\varepsilon el(W)$, $Kl_sW\varepsilon el(X)$, some $s<1$.

Now, we check that: $Kl_sW = int(Kl_sW)$; we consider P with $P\ \varepsilon\ el(Kl_sW)$. There exist R,Q with $R\varepsilon el(P)$, $R\varepsilon el(Q)$, $Q\varepsilon\mu_sW$. Then $R\varepsilon\mu_sW$ hence $R\varepsilon el(kl_sW)$ so $Kl_tR\varepsilon Kl_sW$ for some $t<1$ and finally $R\ \varepsilon\ el(int(Kl_sW))$. By the inference rule 4.1, $Kl_sW\varepsilon el(int(Kl_sW))$ and the identity follows.

Now, as $Kl_sW\varepsilon el(X)$, we have $int(Kl_sW)\varepsilon el(int(X))$ hence $Kl_sW\ \varepsilon el(int(X))$ and thus $Kl_tU\varepsilon el(int(X))$ for some $t<1$ so, finally, $U\varepsilon el(int(int(X)))$ and the inference rule shows that $int(X)\varepsilon el(int(int(X)))$. □

Corollary 7.2. *The rough mereological topology induced by the rough inclusion $\mu_{\top,r}$ is a quasi–topology.*

8. Mereogeometry

Predicates μ_r may be regarded as weak metrics also in the context of geometry. From this point of view, we may apply μ in order to define basic notions of rough mereological geometry.

In the language of this geometry, we may approximately describe and approach geometry of objects described by data tables; a usage for this geometry may be found e.g. in navigation and control tasks of mobile robotics [1], [3], [23], [39], [43], [44].

It is well-known (cf. [90], [9]) that the geometry of Euclidean spaces may be based on some postulates about the basic notions of a point and the ternary equi–distance functor. In [90] postulates for Euclidean geometry over a real–closed field were given based on the functor of betweenness and the quaternary equi–distance functor. Similarly, in [9], a set of postulates aimed at rendering general geometric features of geometry of finite–dimensional spaces over reals has been discussed, the primitive notion there being that of nearness.

Geometrical notions have been applied in e. g. studies of semantics of spatial prepositions [6] and in inferences via cardinal directions cf. e.g [45].

It may not be expected that a geometry induced from a rough mereological context proves to be a Euclidean one, however, we demonstrate that we may introduce in the rough mereological context functors of nearness, betweenness and equidistance that satisfy basic postulates about these functors valid in Euclidean spaces.

8.1. Rough mereological distance, betweenness

We first introduce a notion of distance κ_r in our rough mereological universe by letting

$$\kappa_r(X,Y) \Longleftrightarrow r = \min\{u,w : X\varepsilon\mu_u^+(Y) \wedge Y\varepsilon\mu_w^+(X)\}.$$

We now introduce the notion of betweenness as a functor $T(X,Y)$ of two individual names; the statement $Z\varepsilon T(X,Y)$ reads as 'Z *is between* X *and* Y:

$$Z\varepsilon T(X,Y) \Longleftrightarrow Z\varepsilon Z \wedge \forall W.\kappa_r(Z,W) \wedge \kappa_s(X,W) \wedge \kappa_t(Y,W) \Longrightarrow s \leq r \leq t \vee t \leq r \leq s.$$

Thus, $Z\varepsilon T(X,Y)$ holds when the rough mereological distance κ between Z and W is in the non–oriented interval (i.e. between) [distance of X to W, distance of Y to W] for any W.

We check that T satisfies the axioms of Tarski [90] for *betweenness*.

Proposition 8.1. *The following properties hold:*

1. $Z\varepsilon T(X,X) \Longrightarrow Z = X$ *(identity);*
2. $Y\varepsilon T(X,U) \wedge Z\varepsilon T(Y,U) \Longrightarrow Y\varepsilon T(X,Z)$ *(transitivity);*
3. $Y\varepsilon T(X,Z) \wedge Y\varepsilon T(X,U) \wedge X \neq Y \Longrightarrow Z\varepsilon T(X,U) \vee U\varepsilon T(X,Z)$ *(connectivity).*

Proof:
By means of κ, the properties of betweenness in our context are translated into properties of betweenness in the real line which hold by the Tarski theorem [90], Theorem 1. □

8.2. Nearness

We may also apply κ to define in our context the functor N of nearness proposed in van Benthem [9]:

$$Z\varepsilon N(X,Y) \Longleftrightarrow Z\varepsilon Z \wedge (\kappa_r(Z,X) \wedge \kappa_s(X,Y) \Longrightarrow s < r).$$

Here, nearness means that Z is closer to X than to Y (recall that rough mereological distance is defined in an opposite way: the smaller r, the greater distance).

Then the following hold i.e. N does satisfy all axioms for nearness in [9].

Proposition 8.2.
1. $Z\varepsilon N(X,Y) \wedge Y\varepsilon N(X,W) \Longrightarrow Z\varepsilon N(X,W)$ *(transitivity)*;
2. $Z\varepsilon N(X,Y) \wedge X\varepsilon N(Y,Z) \Longrightarrow X\varepsilon N(Z,Y)$ *(triangle inequality)*;
3. *non*$(Z\varepsilon N(X,Z))$ *(irreflexivity)*;
4. $Z = X \vee Z\varepsilon N(Z,X)$ *(selfishness)*;
5. $Z\varepsilon N(X,Y) \Longrightarrow Z\varepsilon N(X,W) \vee W\varepsilon N(X,Y)$ *(connectedness).*

Proof:
(4) follows by (RM1); (3) is obvious. In proofs of the remaining properties, we introduce a symbol $\mu(X,Y)$ as a value of r for which $\kappa_r(X,Y)$. Then, for (1), assume that $Z\varepsilon N(X,Y), Y\varepsilon N(X,W)$ i.e. $\mu(Z,X) > \mu(X,Y), \mu(X,Y) > \mu(X,W)$ hence $\mu(Z,X) > \mu(X,W)$ i.e. $Z\varepsilon N(X,W)$. In case (2), $Z\varepsilon N(X,Y)$, $X\varepsilon N(Y,Z)$ mean $\mu(Z,X) > \mu(X,Y)$, $\mu(X,Y) > \mu(Y,Z)$ so $\mu(Z,X) > \mu(Y,Z)$ i.e. $X\varepsilon N(Z,Y)$. Concerning (v), $Z\varepsilon N(X,Y)$ implies that $\mu(Z,X) > \mu(X,Y)$ hence either $\mu(Z,X) > \mu(X,W)$ meaning $Z\varepsilon N(X,W)$ or $\mu(X,W) > \mu(X,Y)$ implying $W\varepsilon N(X,Y)$. □

We now may introduce the notion of equi–distance in the guise of either a functor $Eq(X,Y)$ or a functor $D(X,Y,Z,W)$ defined as follows:

$$Z\varepsilon Eq(X,Y) \Longleftrightarrow Z\varepsilon Z \wedge (non(X\varepsilon N(Z,Y)) \wedge non(Y\varepsilon N(Z,X))).$$

It follows that

Proposition 8.3. $Z\varepsilon Eq(X,Y) \Longleftrightarrow Z\varepsilon Z \wedge (\forall r.\kappa_r(X,Z) \Longleftrightarrow \kappa_r(Y,Z))$.

We may define a functor of equi–distance following Tarski [90]:

$$D(X,Y,Z,W) \Longleftrightarrow (\forall r.\kappa_r(X,Y) \Longleftrightarrow \kappa_r(Z,W)).$$

These functors do clearly satisfy the following (cf. [9], [90]).

Proposition 8.4. *1.* $Z\varepsilon Eq(X,Y) \wedge X\varepsilon Eq(Y,Z) \Longrightarrow Y\varepsilon Eq(Z,X)$ *(triangle equality);*

2. $Z\varepsilon T(X,Y) \wedge W\varepsilon Eq(X,Y) \Longrightarrow D(Z,W,X,W)$ *(circle property);*

3. $D(X,Y,Y,X)$ *(reflexivity);*

4. $D(X,Y,Z,Z) \Longrightarrow X = Y$ *(identity);*

5. $D(X,Y,Z,U) \wedge D(X,Y,V,W) \Longrightarrow D(Z,U,V,W)$ *(transitivity).*

One may also follow van Benthem's proposal for a betweenness functor defined via the nearness functor as follows:

$$Z\varepsilon T_B(X,Y) \Longleftrightarrow (\forall W.Z\varepsilon W \vee Z\varepsilon N(X,W) \vee Z\varepsilon N(Y,W).$$

One checks in a straightforward way that

Proposition 8.5. *The functor T_B of betweenness defined according to the above does satisfy the Tarski axioms.*

8.3. Points

The notion of a point may be introduced in a few ways; e.g. following Tarski [88], one may introduce points as classes of names forming ultrafilters under the ordering induced by the functor of being an element el. Another way, suitable in practical cases, where the universe, or more generally, each ultrafilter F as above is finite i.e. principal (meaning that there exists an object X such that F consists of those Y's for which $X\varepsilon el(Y)$ holds) is to define points as atoms of our universe under the functor of being an element i.e. we define a constant name AT as follows:

$$X\varepsilon AT \Longleftrightarrow X\varepsilon X \wedge non(\exists Y.Y\varepsilon el(X) \wedge non(X\varepsilon = (Y)).$$

We will refer to such points as to *atomic points.* We adopt here this notion of a point.

Clearly, restricting ourselves to atomic points, we preserve all properties of functors of betweenness, nearness and equi–distance proved above to be valid in the universe V.

9. Examples

In this section, we will give some examples related to notions and applications thereof presented in the preceding sections.

Our universe will be selected from a quad–tree in the Euclidean plane formed by squares $[k+\frac{i}{2^s}, k+\frac{i+1}{2^s}] \times [l+\frac{j}{2^s}, l+\frac{j+1}{2^s}]$ where $k, l \in Z, i, j = 0, 1, ..., 2^s - 1$ and $s = 0, 1, 2, ...$

The choice of atomic points will depend on the level of granularity of knowledge we assume; we may suppose that our objects are localized in space with a positive degree of uncertainty. We will express this uncertainty assuming that our sensor system perceives each square X as the square X' whose each side length is that of X plus 2α where $\alpha = 2^{-s}$ for some $s > 1$ (we then express uncertainty as uncertainty of location applying "hazing" of objects cf. [92]). By this assumption, we may restrict ourselves to squares with the side length at least 4α (as smaller squares would be localized with uncertainty too high); in consequence, atomic points will be all squares of the above form having the side length equal to 4α. In our example we let for simplicity $4\alpha = 1$. Our atomic points are therefore squares of the form $[k, k+1] \times [l, l+1]$, $k, l \in \mathbf{Z}$.

We will define functors μ_r by letting

$$X \varepsilon \mu_r(Y) \Leftrightarrow \frac{\lambda(X' \cap Y')}{\lambda(X')} \geq r$$

where X', Y' are enlargements of X, Y defined above and λ is the area (Lebesgue) measure in the two-dimensional plane. We may check straightforwardly that

Proposition 9.1. *Functors μ_r satisfy (RM1)-(RM4).*

Let us remark that this measure is a continuous extension of the measure proposed in [63] for the case of discrete information systems (cf. in this respect [62]).

9.1. Mereogeometry

We will adopt the notion of betweenness T_B based on the nearness functor. Then we find that e.g. the following triples (X, Z, Y) do satisfy the formula $Z \varepsilon T_B(X, Y)$:

1. $([0,1] \times [0,1], [1,2] \times [0,1], [2,3] \times [0,1])$;
2. $([0,1] \times [0,1], [0,1] \times [1,2], [0,1] \times [2,3])$;
3. $([0,1] \times [0,1], [1,2] \times [1,2], [2,3] \times [2,3])$;
4. $([2,3] \times [2,3], [1,2] \times [1,2], [0,1] \times [0,1])$;
5. $([0,1] \times [0,1], [1,2] \times [0,1], [2,3] \times [1,2])$.

Clearly, all translates over the digital space $\mathbf{Z}^2$ of the above triples as well as all their rotations by a multiplicity of $\pi/2$ preserve the functor T_B.

The equi–distance functor Eq may be used to define spheres; for instance, admitting as Z the square $[0,1] \times [0,1]$, we have the sphere

$$S(Z; 1/3) = Kl([0,1] \times [1,2], [0,1] \times [-1,0], [1,2] \times [0,1], [-1,0] \times [0,1]\}.$$

A line segment may be defined via the auxiliary notion of a pattern; we introduce this notion as a functor Pt.

We let

$$Pt(X, Y, Z) \Longleftrightarrow Z\varepsilon T_B(X, Y) \vee X\varepsilon T_B(Z, Y) \vee Y\varepsilon T_B(X, Z).$$

We will say that a finite sequence $X_1, X_2, ..., X_n$ of points belong in a line segment whenever $Pt(X_i, X_{i+1}, X_{i+2})$ for $i = 1, , ..., n-2$; formally, we introduce the functor $Line$ of finite arity defined via

$$Line(X_1, X_2, ..., X_n) \Longleftrightarrow \forall i < n-1.Pt(X_i, X_{i+1}, X_{i+2})$$

and then we let

$$Line\_seg(X_1, X_2, ..., X_n)\varepsilon Kl(X_1, X_2, ..., X_n : Line(X_1, X_2, ..., X_n)).$$

In particular, classes of sequences:

1. $([0,1] \times [i, i+1])_i$;
2. $([i, i+1] \times [0,1])_i$;
3. $([i, i+1] \times [i, i+1])_i$;
4. $([i, i+1] \times [-i, -i-1])_i$

for $i \in [-n, n]$ where $n = 1, 2, ...,$ are line segments.

It is clearly possible to introduce line segments of various types by means of specialized pattern functors.

The notion of orthogonality may be introduced in a well-known way; we introduce a functor $Ortho$: for two line segments A, B, with $Z\varepsilon el(A)$, $Z\varepsilon el(B)$, we let

$$Ortho(A, B) \Longleftrightarrow \exists X, Y, U, W.X, Y\varepsilon el(A) \wedge U, W\varepsilon el(B) \wedge non(X\varepsilon = (Y)) \wedge$$
$$non(U\varepsilon = (W)) \wedge U\varepsilon Eq(X, Y) \wedge W\varepsilon Eq(X, Y)$$

(read:A, B are orthogonal). In particular, line segments being classes of sets

1. $([0,1] \times [i, i+1])_i$;
2. $([i, i+1] \times [0,1])_i$;

   as well as classes of

3. $([i, i+1] \times [i, i+1])_i$;
4. $([i, i+1] \times [-i, -i-1])_i$.

are orthogonal.

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