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A Comparative Study of Some Generalized Rough Approximations

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Abstract. In this paper we focus upon a comparison of some generalized rough approximations of sets, where the classical indiscernibility relation is generalized to any binary reflexive relation. We aim at finding the best of several candidates for generalized rough approximation mappings, where both definability of sets by elementary granules of information as well as the issue of distinction among positive, negative, and border regions of a set are taken into account.

Keywords: granulation of information, approximation space, rough approximation

To Grzegorz and Joanna

1. Introduction

Until now the question of generalization of Pawlak's rough approximation of sets has gained quite much interest in the rough-set community [3, 4, 8, 10, 11, 13, 14, 15, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 39, 38]. What a motivation might be then to discuss this problem again? Doing a work on modelling of some concepts of granular nature, I was stuck when trying to represent the modelled concepts by their lower and upper approximations. It was simply because several competitive approximation mappings

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were recommended in the literature. In the present paper we focus upon simple-to-grasp extensions of the notion of rough approximation, having in mind possible applications to the socially embedded game theory [1, 2]. Interesting, yet more sophisticated generalizations like [3, 4, 8, 10, 14, 15, 28, 29, 30, 35, 39] will not be discussed in detail here.

Thus, in this paper we study and compare only some generalized rough approximation mappings. The starting point is a generalized approximation space [18, 27, 32], where the uncertainty mapping generates a covering which nevertheless needs not to be a partition of the universe. Elements of the covering are elementary granules of information associated with the approximation space. The notion of granulation of information was introduced by Zadeh [36]. Rephrasing Zadeh's definition in accordance with suggestions of Lin [16], a *granule* is a clump of objects of some class, drawn together or towards some object by indistinguishability, similarity or functionality.

The notion of approximation space is a generalization of the notion of *information system* in the sense of Pawlak [20]. In information systems, and also in approximation spaces, any concept which may be represented in the form of a set x of objects of the universe is approximated by a pair of exact sets of objects, called the *lower* (or *inner*) and *upper (outer) approximations* of x. A set x, and hence the concept represented by x, is called *rough* in case its lower and upper approximations are different. Rough set methodology [13, 20, 24] is a methodology for handling uncertainty arising from granularity of information in the domain of discourse. The primarily considered source of granulation of information was indiscernibility between objects in a set. Later the classical indiscernibility-based approach was extended to capture similarity-based granulation [14, 15, 18, 27, 28, 29, 30, 32]. Thus, the classical notion of rough approximation of sets evolved to more general ones.[1] Admittance of uncertainty mappings, generating arbitrary coverings which need not to be partitions of the universe, opens new possibilities in searching for an appropriate notion of generalized approximation of sets; it also increases difficulties.

The paper is organized as follows. In Sect. 2 a general notion of approximation space is presented. In the next section we briefly recall the classical notion of rough approximation of sets. In Sect. 4 we consider several mappings to approximate sets of objects along the lines of Pawlak's approach. Some of them are well-known in the literature, some were proposed earlier but have not gained much interest, some are new. We define a list of "rationality" postulates that seem to be natural and reasonable requirements that approximation mappings should satisfy. In the next step we investigate properties of the defined mappings and test the results against the postulates. Some related works are concisely discussed in Sect. 5. Section 6 contains a brief summary.

Throughout the paper, the cardinality of a set x will be denoted by $\#x$, the power set of x by $\wp(x)$, and the set-theoretical complement of x by $-x$. Given a binary relation (in particular, a mapping) $\rho \subseteq x \times y$ and sets $x_0 \subseteq x$ and $y_0 \subseteq y$, by $\rho^{\rightarrow}(x_0)$ and $\rho^{\leftarrow}(y_0)$ we shall denote the image of x_0 and the converse image of y_0 given by ρ.

Let U be a non-empty set. Henceforth we shall consider mappings (operators) $f : \wp(U) \mapsto \wp(U)$. We can define a partial ordering relation, $\leq$, on the set of all such mappings as follows: $f \leq g$ iff $\forall x \subseteq U. f(x) \subseteq g(x)$, for every $f, g : \wp(U) \mapsto \wp(U)$. By id we denote the identity mapping on $\wp(U)$. According to our notation, $g \circ f : \wp(U) \mapsto \wp(U)$ defined along the standard lines by $(g \circ f)(x) = g(f(x))$ for every $x \subseteq U$, is a composition of f and g. We call g *dual* to f, written $g = f^d$, if $g(x) = -f(-x)$ for each $x \subseteq U$. f is called *decreasing* if $f \leq$ id, i.e., $f(x) \subseteq x$ for each $x \subseteq U$. Similarly, f is *increasing* if id $\leq f$, i.e., $x \subseteq f(x)$ for each $x \subseteq U$. The mapping f is *monotone* iff for every $x, y \subseteq U$,

[1] See, e.g., [35] for an interesting survey of various rough set models.

$x \subseteq y$ implies $f(x) \subseteq f(y)$. Finally, f is *idempotent* (respectively, *co-idempotent*) iff $f \circ f \leq f$ (resp., $f \leq f \circ f$). The following observations may be useful.

Proposition 1.1. Let $\pm \in \{\cup, \cap\}$. For any mappings $f, g : \wp(U) \mapsto \wp(U)$ and $x, y \subseteq U$, the following properties hold:

(a) $(f^d)^d = f$.

(b) $(f \circ g)^d = f^d \circ g^d$.

(c) $f(\emptyset) = \emptyset$ iff $f^d(U) = U$.

(d) If f, g are monotone, then $f \circ g$ is monotone.

(e) $f(x \cup y) = f(x) \cup f(y)$ iff $f^d(x \cap y) = f^d(x) \cap f^d(y)$.

(f) If $f(x \pm y) = f(x) \pm f(y)$ and $g(x \pm y) = g(x) \pm g(y)$, then $(f \circ g)(x \pm y) = (f \circ g)(x) \pm (f \circ g)(y)$.

Proof:
We only show (e). The rest is left as an exercise. Assume $f(x \cup y) = f(x) \cup f(y)$. Then $f^d(x \cap y) = -f(-(x \cap y)) = -f(-x \cup -y) = -(f(-x) \cup f(-y)) = -f(-x) \cap -f(-y) = f^d(x) \cap f^d(y)$. Conversely, if $f^d(x \cap y) = f^d(x) \cap f^d(y)$, then $f(x \cup y) = -f^d(-(x \cup y)) = -f^d(-x \cap -y) = -(f^d(-x) \cap f^d(-y)) = -f^d(-x) \cup -f^d(-y) = f(x) \cup f(y)$. □

2. A General Notion of Approximation Space

In this section we briefly recall a general notion of approximation space [18, 27, 32]. By an *approximation space* we mean a triple $\mathcal{A} = (U, I, \kappa)$, where U is a non-empty set called the universe, $I : U \mapsto \wp(U)$ is an uncertainty mapping, and $\kappa : \wp(U) \times \wp(U) \mapsto [0, 1]$ is a rough inclusion function.[2]

The mapping I may be viewed as a granulation function which assigns to each $u \in U$ a set of elements of U drawn towards u, i.e. an elementary granule of information. In this way an indexed family of sets, being elementary granules of information from our perspective, $I^{\rightarrow}(U) = \{I(u) \mid u \in U\}$ is obtained. Formation of (elementary and compound) granules from elements of the universe is an important step in granular computing. In [18, 32] a definition of an uncertainty mapping which may possibly be constructed from given data is proposed.

If $w \in I(u)$ is understood as w is in some sense similar to u, then it seems to be reasonable to assume that

$$\forall u \in U. u \in I(u). \tag{1}$$

Then $I^{\rightarrow}(U)$ is a covering of U, i.e.,

$$U = \bigcup I^{\rightarrow}(U). \tag{2}$$

Henceforth we shall only consider I satisfying (1).[3] The role of the uncertainty mapping I may be played by a binary relation on U. More precisely, any mapping I satisfying (1) generates a reflexive relation

[2] For the sake of simplicity we omit parameters.

[3] In our approach we do not stipulate that similarity is symmetric in general.

$\rho \subseteq U \times U$ such that for every $u, w \in U$,

$$(w,u) \in \rho \text{ iff } w \in I(u). \tag{3}$$

Then $(w,u) \in \rho$ reads as "w is similar to u". Conversely, any reflexive relation $\rho \subseteq U \times U$ generates an uncertainty mapping $I : U \mapsto \wp(U)$, satisfying (1), where

$$I(u) = \rho^{\leftarrow}(\{u\}). \tag{4}$$

It can be useful to denote somehow the set of objects to which $u \in U$ is similar. Thus, let

$$\tau(u) = \rho^{\rightarrow}(\{u\}). \tag{5}$$

Observe that the sets $I(u)$ and $\tau(u)$ are different in general. They are identical if ρ is symmetric. Symmetry of ρ and the corresponding condition for I may be expressed as follows:

$$\begin{aligned} &\forall u, w \in U.((u,w) \in \rho \to (w,u) \in \rho). \\ &\forall u, w \in U.(u \in I(w) \to w \in I(u)). \end{aligned} \tag{6}$$

It is easy to see that the following conditions are equivalent:

$$\begin{aligned} &(i) && \rho \text{ is an equivalence relation on } U. \\ &(ii) && I^{\rightarrow}(U) \text{ is a partition of } U. \\ &(iii) && \forall u, v \in U.(I(u) = I(v) \vee I(u) \cap I(v) = \emptyset). \\ &(iv) && \forall u, v \in U.(u \in I(v) \to I(u) = I(v)). \end{aligned} \tag{7}$$

For any pair (x, y) of subsets of U, the rough inclusion function κ defines the degree of inclusion of x in y. Various rough inclusion functions may be considered [5, 22, 27, 32]. Where U is finite, one well-known rough inclusion function (sometimes called "standard") is defined as follows:

$$\kappa(x,y) \stackrel{\text{def}}{=} \begin{cases} \frac{\#(x \cap y)}{\# x} & \text{if } x \neq \emptyset \\ 1 & \text{otherwise} \end{cases} \tag{8}$$

Let us observe that

$$\begin{aligned} &\kappa(x,y) = 1 \text{ iff } x \subseteq y; \\ &\kappa(x,y) > 0 \text{ iff } x \cap y \neq \emptyset. \end{aligned} \tag{9}$$

$\kappa(x,y) = 1$ means that x is certainly included in y since every element of x belongs to y. This gives rise to the concept of *positive region* of a set $x \subseteq U$ which is defined as the set of all objects $u \in U$ such that $\kappa(I(u), x) = 1$ (or equivalently, $I(u) \subseteq x$). On the other hand, $\kappa(x,y) = 0$ means that x is certainly not included in y as no element of x belongs to y. Hence by the *negative region* of x we understand the set of all $u \in U$ such that $\kappa(I(u), x) = 0$ (i.e., $I(u) \cap x = \emptyset$). Finally, $\kappa(x,y) > 0$ means that x is possibly included in y as at least one element of x belongs to y. Hence the *border region* of x may be defined as the set of all elements $u \in U$ such that $0 < \kappa(I(u), x) < 1$.[4]

[4]In this paper we give the priority to the mapping I. One may also start with the notion of approximation space, where I is substituted by τ. Though it was not checked in detail, the derived results should be analogous.

3. Lower and Upper Rough Approximations of Sets: The Classical Case

Keeping with our notation, let us recall Pawlak's idea of approximation of sets of objects [20, 13] in terms of approximation space. We consider an approximation space $\mathcal{A} = (U, I, \kappa)$ as described above, where U is a finite set of objects, I is an uncertainty mapping such that the family $I^{\rightarrow}(U)$ is a partition of U, and the rough inclusion function κ is defined by (8). For any $u \in U$, all objects belonging to $I(u)$ are indiscernible from u and from one another. Moreover, for any $u, w \in U$, the elementary granules $I(u)$, $I(w)$ are identical or disjoint in accordance with (7). Given a set of objects $x \subseteq U$, its *lower* and *upper approximations*, $\mathrm{LOW}(x)$ and $\mathrm{UPP}(x)$, respectively, are defined as follows:[5]

$$\mathrm{LOW}(x) \stackrel{\text{def}}{=} \{u \in U \mid \kappa(I(u), x) = 1\}. \tag{10}$$

$$\mathrm{UPP}(x) \stackrel{\text{def}}{=} \{u \in U \mid \kappa(I(u), x) > 0\}. \tag{11}$$

Elements of $\mathrm{LOW}(x)$ certainly belong to x, while elements of $\mathrm{UPP}(x)$ possibly belong to x. Thanks to (9), the above equations are, respectively, equivalent with

$$\mathrm{LOW}(x) = \{u \in U \mid I(u) \subseteq x\}; \tag{12}$$

$$\mathrm{UPP}(x) = \{u \in U \mid I(u) \cap x \neq \emptyset\}. \tag{13}$$

It is not difficult to see that the following equations hold in this case as well:

$$\mathrm{LOW}(x) = \bigcup\{I(u) \mid u \in U \wedge I(u) \subseteq x\}. \tag{14}$$

$$\mathrm{UPP}(x) = \bigcup\{I(u) \mid u \in U \wedge I(u) \cap x \neq \emptyset\}. \tag{15}$$

LOW and UPP defined by (10) and (11), respectively, may be viewed as approximation mappings $\mathrm{LOW}, \mathrm{UPP} : \wp(U) \mapsto \wp(U)$, assigning to subsets of U their approximations, i.e. some subsets of U. Let us recall other basic, yet interesting properties of the lower and upper approximations of sets.

Theorem 3.1. For any sets $x, y \subseteq U$, we have that:

(a) $\mathrm{LOW}(x) \subseteq x \subseteq \mathrm{UPP}(x)$.

(b) $\mathrm{LOW}(\emptyset) = \mathrm{UPP}(\emptyset) = \emptyset$.

(c) $\mathrm{LOW}(U) = \mathrm{UPP}(U) = U$.

(d) If $x \subseteq y$, then $\mathrm{LOW}(x) \subseteq \mathrm{LOW}(y)$ and $\mathrm{UPP}(x) \subseteq \mathrm{UPP}(y)$.

(e) $\mathrm{LOW}(x \cup y) \supseteq \mathrm{LOW}(x) \cup \mathrm{LOW}(y)$.

(f) $\mathrm{LOW}(x \cap y) = \mathrm{LOW}(x) \cap \mathrm{LOW}(y)$.

(g) $\mathrm{UPP}(x \cup y) = \mathrm{UPP}(x) \cup \mathrm{UPP}(y)$.

(h) $\mathrm{UPP}(x \cap y) \subseteq \mathrm{UPP}(x) \cap \mathrm{UPP}(y)$.

(i) $\mathrm{LOW}(x) = -\mathrm{UPP}(-x)$.

(j) $\mathrm{UPP}(x) = -\mathrm{LOW}(-x)$.

(k) $\mathrm{LOW}(\mathrm{LOW}(x)) = \mathrm{UPP}(\mathrm{LOW}(x)) = \mathrm{LOW}(x)$.

(l) $\mathrm{UPP}(\mathrm{UPP}(x)) = \mathrm{LOW}(\mathrm{UPP}(x)) = \mathrm{UPP}(x)$.

[5] We omit the reference to $\mathcal{A}$ for simplicity.

Thus, the lower and upper approximation operations LOW and UPP, respectively, are dual (see (i), (j)) and, moreover, satisfy the usual conditions for the interior and closure operations, respectively. Along the standard lines (see, e.g., [6]), every set $x \subseteq U$ such that $\mathrm{UPP}(x) = x$ is closed. Then as a topology $O \subseteq \wp(U)$ on U we may take the family of complements of such closed sets, i.e.,

$$O \stackrel{\text{def}}{=} \{U - x \mid x \subseteq U \wedge \mathrm{UPP}(x) = x\}. \tag{16}$$

Obviously, every set $x \subseteq U$ such that $\mathrm{LOW}(x) = x$ is open. In fact, $\mathrm{LOW}(x)$ and $\mathrm{UPP}(x)$ are closed and open (i.e., clopen) by (k), (l). Notice that the topology O is generated by the uncertainty mapping I. The topological space (U, O) is not even $\mathcal{T}_0$. To see this, consider two different $u, v \in U$ that are indiscernible, i.e., $I(u) = I(v)$. We show that for every open set $x \in O$, $u \notin x$ iff $v \notin x$. Let $x \in O$. Then $x = U - y$ where $\mathrm{UPP}(y) = y$. We have that $u \notin x$ iff $u \in \mathrm{UPP}(y)$ iff $I(u) \cap y \neq \emptyset$ iff $I(v) \cap y \neq \emptyset$ iff $v \in \mathrm{UPP}(y)$ iff $v \notin x$. Hence there is no open set separating u and v. Nevertheless, the topological spaces with topologies, generated by relations of indiscernibility of objects with respect to attributes in Pawlak's information systems[6], are metrizable and complete [21].

A set x is called *exact* if $\mathrm{LOW}(x) = \mathrm{UPP}(x)$; otherwise it is a *rough set*. Thus, every exact set is clopen. Rough sets may be represented by pairs of exact sets, viz. their lower and upper approximations.

The notion of definability of a set of objects of the universe in an approximation space may be defined in various ways. We understand definability of a set as the possibility to represent the set as a union of elementary granules of the approximation space. More precisely, $x \subseteq U$ is *definable* in an approximation space $\mathcal{A}$ (or simply definable if $\mathcal{A}$ is known) iff there is a set $y \subseteq U$ such that $x = \bigcup\{I(u) \mid u \in y\}$ (i.e., $x = \bigcup I^{\rightarrow}(y)$). It turns out that a set is definable iff it is exact. In detail, for any set $x \subseteq U$, it holds that

$$\exists y \subseteq U. x = \bigcup I^{\rightarrow}(y) \text{ iff } \mathrm{LOW}(x) = \mathrm{UPP}(x). \tag{17}$$

4. Some Extensions and Generalizations

The classical notion of rough approximation has been generalized in several directions (see, e.g., [3, 4, 8, 10, 11, 13, 14, 15, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 39, 40, 38]). In this paper we are interested in generalization of the classical case to that one, where the uncertainty mapping I associated with an approximation space $\mathcal{A} = (U, I, \kappa)$ satisfies (1) but $I^{\rightarrow}(U)$ may or may not be a partition of U. Also (6) is not required. The universe U is a non-empty set, not necessarily finite. We assume that κ satisfies (9). In [3, 4, 26, 33, 34, 38] topological or algebraic aspects dominate, while the apects considered in [11, 14] (and also in our paper) are definability of sets by elementary granules of information and distinction among positive, negative, and border regions of a set.

4.1. Postulates for rough approximation mappings

First, let us determine what fundamental properties any reasonable rough approximation mapping $f : \wp(U) \mapsto \wp(U)$, in an approximation space $\mathcal{A}$ as above, should possibly possess. We distinguish two kinds of approximation mappings: lower and upper approximation mappings (in short low- and upp-mappings). Such "rationality" postulates for low- and upp-mappings could have the following forms:

[6]These systems [20] are particular cases of approximation spaces as above.

(a1) Every low-mapping f is decreasing (i.e., $f \leq \mathrm{id}$).

(a2) Every upp-mapping f is increasing (i.e., $\mathrm{id} \leq f$).

(a3) If f is a low-mapping, then $(*) \forall x \subseteq U. \forall u \in f(x). \kappa(I(u), x) = 1$ (i.e., $I(u) \subseteq x$).

(a4) If f is an upp-mapping, then $(**) \forall x \subseteq U. \forall u \in f(x). \kappa(I(u), x) > 0$ (i.e., $I(u) \cap x \neq \emptyset$).

(a5) For each $x \subseteq U$, $f(x)$ is definable in $\mathcal{A}$.

(a6) For each $x \subseteq U$ definable in $\mathcal{A}$, $f(x) = x$.

The motivation behind (a1) and (a2) is clear. The postulates (a3) and (a4) are useful, e.g. in the case of classification of objects. Then it may be of great importance to be able to distinguish the positive, negative, and border regions of a set. In detail, (a3) says that the approximation of x by means of any low-mapping is a subset of the positive region of x. On the other hand, (a4) stipulates the approximation of x by means of any upp-mapping to be disjoint with the negative region of x. The postulate (a5) is important if the domain of discourse is viewed from the perspective of granulation of information. Elementary granules of information are then the least blocks at our disposal to form more complex structures representing concepts. The postulate (a6), relating definability and approximation, says that the sets that are definable need no approximation. Notice that (a5) and (a6) imply $f \circ f = f$. One could also formulate other properties, e.g., low- and upp-mappings to be topological interior and closure operations, respectively. Given appropriate low- and upp-mappings, say f and g, respectively, each set $x \subseteq U$ can be represented by the pair $(f(x), g(x))$. Then one could claim f and g to be dual. However, as we shall see later, finding appropriate candidates for low- and upp-mappings, satisfying even those six conditions above, is not an easy matter in the general case.

4.2. Basic approximation mappings and their properties

First recall that $u \in \tau(w)$ iff $w \in I(u)$ and that $f \leq g$ iff $f(x) \subseteq g(x)$ for each $x \subseteq U$. In this section we consider some mappings $f_0, f_1 : \wp(U) \mapsto \wp(U)$, where for each $x \subseteq U$,

$$\begin{aligned} f_0(x) &\stackrel{\mathrm{def}}{=} \{u \in U \mid \tau(u) \cap x \neq \emptyset\}, \\ f_1(x) &\stackrel{\mathrm{def}}{=} \{u \in U \mid I(u) \cap x \neq \emptyset\}, \end{aligned} \tag{18}$$

and their dual mappings f_0^d, f_1^d. Observe that

$$\begin{aligned} f_0(x) &= \bigcup I^{\rightarrow}(x) = \bigcup \{I(u) \mid u \in x\}, \\ f_0^d(x) &= \{u \in U \mid \tau(u) \subseteq x\}, \\ f_1^d(x) &= \{u \in U \mid I(u) \subseteq x\}. \end{aligned} \tag{19}$$

These mappings are of particular importance as we shall see later on. f_1 and f_1^d are the classical rough upper and lower approximation mappings, respectively, adapted to the general case. Theorem 3.1 holds regardless the cardinality of U. Thus, if $I^{\rightarrow}(U)$ is a partition of U, f_1 and f_1^d enjoy all the properties of UPP and LOW, respectively, presented in Sect. 3. The mappings f_0 and f_0^d were considered, e.g., in [11, 14, 15, 28, 29, 30], and f_0 additionally in [33]. The following properties can be proved.[7]

[7]For obvious reason, some of them can be found in the literature.

Theorem 4.1. For any sets $x, y \subseteq U$, objects $u, w \in U$, and $i = 0, 1$, it holds that:

$$
\begin{array}{ll}
(a) & f_0^d \leq \mathrm{id} \leq f_0. \\
(b) & f_1^d \leq \mathrm{id} \leq f_1. \\
(c) & f_0(x) \text{ is definable.} \\
(d) & \forall u \in f_1(x).\kappa(I(u), x) > 0. \\
(e) & \forall u \in f_1^d(x).\kappa(I(u), x) = 1. \\
(f) & \text{If } \tau(u) = \tau(w), \text{ then } u \in f_0(x) \text{ iff } w \in f_0(x); \text{ and similarly for } f_0^d. \\
(g) & \text{If } I(u) = I(w), \text{ then } u \in f_1(x) \text{ iff } w \in f_1(x); \text{ and similarly for } f_1^d. \\
(h) & f_i(\emptyset) = \emptyset \text{ and } f_i(U) = U; \text{ and similarly for } f_i^d. \\
(i) & f_i \text{ and } f_i^d \text{ are monotone.} \\
(j) & f_i(x \cup y) = f_i(x) \cup f_i(y). \\
(k) & f_i^d(x \cup y) \supseteq f_i^d(x) \cup f_i^d(y). \\
(l) & f_i(x \cap y) \subseteq f_i(x) \cap f_i(y). \\
(m) & f_i^d(x \cap y) = f_i^d(x) \cap f_i^d(y).
\end{array}
$$

Proof:
We only prove (g) and (j) for $i = 0$, leaving the remaining cases as exercises. (g) follows simply by the observation that $I(u) = I(w)$ implies

$$I(u) \cap x \neq \emptyset \text{ iff } I(w) \cap x \neq \emptyset \text{ and } I(u) \subseteq x \text{ iff } I(w) \subseteq x.$$

For (j) notice that $I^{\rightarrow}(x \cup y) = I^{\rightarrow}(x) \cup I^{\rightarrow}(y)$ and apply the definition. □

Let us note that a set $x \subseteq U$ is definable in an approximation space $\mathcal{A}$ iff there is $y \subseteq U$ such that $x = f_0(y)$ or, in other words, iff $x \in f_0^{\rightarrow}(\wp(U))$. The following observations may be useful as well.

Proposition 4.1. Consider any $f : \wp(U) \mapsto \wp(U)$. (a) $f(x)$ is definable for any $x \subseteq U$ iff there is a mapping $g : \wp(U) \mapsto \wp(U)$ such that $f = f_0 \circ g$. (b) The condition $(*)$ (see the postulate (a3)) is satisfied iff $f \leq f_1^d$. (c) The condition $(**)$ is satisfied iff $f \leq f_1$.

Proof:
We only show (a). ($\Rightarrow$) Assume definability of $f(x)$ for any $x \subseteq U$. Hence there is $y \subseteq U$ such that $f(x) = \bigcup\{I(u) \mid u \in y\}$, i.e., $f(x) = f_0(y)$. With every $x \subseteq U$ we can associate a non-empty family $\mathcal{X} = \{y \subseteq U \mid f(x) = f_0(y)\}$. Define a mapping $g : \wp(U) \mapsto \wp(U)$ in such a way that for $x \subseteq U$, $g(x) \in \mathcal{X}$. The existence of g is guaranteed by the axiom of choice. Obviously, $f = f_0 \circ g$. ($\Leftarrow$) Now if $f = f_0 \circ g$ for some $g : \wp(U) \mapsto \wp(U)$, then for each $x \subseteq U$, $f(x) = f_0(g(x))$ as required. □

Hence it follows that $\kappa(I(u), x) > 0$ for each $u \in f_0^d(x)$. Moreover, for each $u \in f_0(x)$, $\kappa(I(u), x) > 0$ if (6) holds. Indeed, we can prove the following result.

Proposition 4.2. If (6) is satisfied (i.e., similarity is symmetric), then $f_0 = f_1$.

Proof:
Assume (6). Consider any $x \subseteq U$ and $u \in U$. Then $u \in f_0(x)$ iff $\exists w \in x.u \in I(w)$ iff (by assumption) $\exists w \in x.w \in I(u)$ iff $I(u) \cap x \neq \emptyset$ iff $u \in f_1(x)$. □

Directly from Theorem 4.1 it follows that f_i is co-idempotent and f_i^d is idempotent ($i = 0, 1$). Moreover, the following characterization can be obtained.

Theorem 4.2. The following conditions are equivalent:

$$
\begin{array}{ll}
(a) & f_0 \circ f_0 = f_0. \\
(b) & f_0^d \circ f_0^d = f_0^d. \\
(c) & I^{\rightarrow}(U) \text{ is a partition of } U.
\end{array}
$$

Proof:
(a) and (b) are equivalent by Proposition 1.1. In virtue of Theorem 4.1, to prove the equivalence of (a) and (c), it suffices to show that (c) holds iff $\forall x \subseteq U.(f_0 \circ f_0)(x) \subseteq f_0(x)$. ($\Rightarrow$) Consider any $x \subseteq U$ and $u \in U$, and assume that $I^{\rightarrow}(U)$ is a partition of U. Let $u \in (f_0 \circ f_0)(x)$. By definition there is $v \in f_0(x)$ such that $u \in I(v)$. Hence there is $w \in x$ such that $v \in I(w)$ and $u \in I(v)$. By assumption in virtue of (7), $I(v) = I(w)$. Thus $u \in f_0(x)$. ($\Leftarrow$) Now assume by contraposition that $I^{\rightarrow}(U)$ is not a partition of U. Then there are $u, v \in U$ such that (A) $u \in I(v)$ and $I(u) \neq I(v)$. Without loss of generality we may assume that there is (B) $w \in I(u)$ such that (C) $w \notin I(v)$. Observe that $f_0(\{v\}) = I(v)$ and $(f_0 \circ f_0)(\{v\}) = f_0(I(v)) = \bigcup\{I(v') \mid v' \in I(v)\}$. Clearly, $w \in (f_0 \circ f_0)(\{v\})$ by (A), (B), while $w \notin f_0(\{v\})$ by (C). In other words, $(f_0 \circ f_0)(\{v\}) \not\subseteq f_0(\{v\})$. □

At the end of this section we generalize the properties (f) and (g) from Theorem 4.1.

Proposition 4.3. Let $g : \wp(U) \mapsto \wp(U)$ be any mapping and $u, w \in U$. (a) If $\tau(u) = \tau(w)$, then for each $x \subseteq U$, $u \in (f_0 \circ g)(x)$ iff $w \in (f_0 \circ g)(x)$; and similarly for f_0^d. (b) If $I(u) = I(w)$, then for each $x \subseteq U$, $u \in (f_1 \circ g)(x)$ iff $w \in (f_1 \circ g)(x)$; and similarly for f_1^d.

4.3. Approximation mappings and their properties: A continuation

In view of the previous results and in accordance with the postulates, any low- or upp-mapping should have the form $f_0 \circ g$, where $g : \wp(U) \mapsto \wp(U)$ satisfies $f_0 \circ g \circ f_0 = f_0$ and, moreover, $f_0 \circ g \leq f_1^d$ in the "lower" case, while $\mathrm{id} \leq f_0 \circ g \leq f_1$ in the "upper" case. Clearly, $\leq$-maximal among the low-mappings and $\leq$-minimal among the upp-mappings would be the best approximators. The greatest element among the low-mappings just described is the mapping $h : \wp(U) \mapsto \wp(U)$ where for any $x \subseteq U$,

$$
\begin{aligned}
h(x) \stackrel{\text{def}}{=} & \bigcup\{(f_0 \circ g)(x) \mid g : \wp(U) \mapsto \wp(U) \\
& \wedge f_0 \circ g \circ f_0 = f_0 \wedge f_0 \circ g \leq f_1^d\}.
\end{aligned} \tag{20}
$$

One may check that h indeed is a low-mapping. Unfortunaly in case U were a finite, yet large set, it would be difficult to obtain h in practice. Let us observe that an analogous construction (using $\bigcap$) does not provide us with the least element of the family of upp-mappings.

Therefore from f_0, f_1 and their dual mappings by means of operations of composition and duality, we define several interesting mappings, being possible candidates for approximation mappings. Thus, let us consider the following mappings $f_i : \wp(U) \mapsto \wp(U)$ ($i = 2, \ldots, 9$).

$$
\begin{aligned}
f_2 &\stackrel{\text{def}}{=} f_0 \circ f_1^d, \text{ i.e., } f_2(x) = \bigcup\{I(u) \mid u \in U \wedge I(u) \subseteq x\}.\\
f_3 &\stackrel{\text{def}}{=} f_0 \circ f_1, \text{ i.e., } f_3(x) = \bigcup\{I(u) \mid u \in U \wedge I(u) \cap x \neq \emptyset\}.\\
f_4 &\stackrel{\text{def}}{=} f_2^d, \text{ i.e., } f_4 = f_0^d \circ f_1, \text{ i.e.,}\\
f_4(x) &= \{u \in U \mid \forall w \in \tau(u).I(w) \cap x \neq \emptyset\}.\\
f_5 &\stackrel{\text{def}}{=} f_3^d, \text{ i.e., } f_5 = f_0^d \circ f_1^d, \text{ i.e.,}\\
f_5(x) &= \{u \in U \mid \forall w \in \tau(u).I(w) \subseteq x\}.\\
f_6 &\stackrel{\text{def}}{=} f_1^d \circ f_1^d, \text{ i.e., } f_6(x) = \{u \in U \mid \forall w \in I(u).I(w) \subseteq x\}.\\
f_7 &\stackrel{\text{def}}{=} f_0 \circ f_6, \text{ i.e., } f_7 = f_0 \circ f_1^d \circ f_1^d = f_2 \circ f_1^d, \text{ i.e.,}\\
f_7(x) &= \bigcup\{I(u) \mid u \in U \wedge \forall w \in I(u).I(w) \subseteq x\}.\\
f_8 &\stackrel{\text{def}}{=} f_1^d \circ f_1, \text{ i.e., } f_8(x) = \{u \in U \mid \forall w \in I(u).I(w) \cap x \neq \emptyset\}.\\
f_9 &\stackrel{\text{def}}{=} f_0 \circ f_8, \text{ i.e., } f_9 = f_0 \circ f_1^d \circ f_1 = f_2 \circ f_1, \text{ i.e.,}\\
f_9(x) &= \bigcup\{I(u) \mid u \in U \wedge \forall w \in I(u).I(w) \cap x \neq \emptyset\}. \qquad (21)
\end{aligned}
$$

Mappings f_2 and f_3 are well-known from the literature (cf. [3, 4, 11, 13, 26, 33, 34]), mappings f_4 and f_5 are considered in [11, 26] and mentioned in [13]. On the other hand, the mappings f_i ($i = 6, \ldots, 9$) are new – at least as far as it was possible to check.

Let us observe the following dependencies. The symbol $\leq$ in the properties below cannot be reversed in general.

Theorem 4.3. For any set $x \subseteq U$ and objects $u, w \in U$, we have that:

(a) $f_5 \leq f_1^d \leq f_2 \leq \text{id} \leq f_4 \leq f_1 \leq f_3$.
(b) $f_5 \leq f_0^d \leq \text{id} \leq f_0 \leq f_3$.
(c) $f_6 \leq f_7 \leq f_1^d$.
(d) $f_8 \leq f_9 \leq f_1$.
(e) $f_i(x)$ is definable if $i = 2, 3, 7, 9$.
(f) $\forall u \in f_i(x).\kappa(I(u), x) = 1$ if $i = 5, 6, 7$.
(g) $\forall u \in f_i(x).\kappa(I(u), x) > 0$ if $i = 2, 4, \ldots, 9$.
(h) If $\tau(u) = \tau(w)$ and $i = 2, 3, 4, 5, 7, 9,$ then $u \in f_i(x)$ iff $w \in f_i(x)$.
(i) If $I(u) = I(w)$ and $i = 6, 8,$ then $u \in f_i(x)$ iff $w \in f_i(x)$.

Proof:
In the proof we use, concisely speaking, the properties obtained in the preceding section. Now we only show that $f_7 \leq f_1^d$, leaving the rest as an exercise. Thus let x be any subset of U and $u \in f_7(x)$. By

definition there is $w \in U$ such that $u \in I(w)$ and for each $v \in I(w)$, $I(v) \subseteq x$. In particular, $u \in I(w)$ and hence $I(u) \subseteq x$. By definition $u \in f_1^d(x)$. □

Several other results are given below.

Theorem 4.4. For any sets $x, y \subseteq U$, we have that:

(a) $f_i(\emptyset) = \emptyset$ and $f_i(U) = U$ for $i = 2, \ldots, 9$.

(b) f_i are monotone for $i = 2, \ldots, 9$.

(c) $f_3(x \cup y) = f_3(x) \cup f_3(y)$.

(d) $f_i(x \cup y) \supseteq f_i(x) \cup f_i(y)$ if $i = 2, 4, \ldots, 9$.

(e) $f_i(x \cap y) = f_i(x) \cap f_i(y)$ if $i = 5, 6$.

(f) $f_i(x \cap y) \subseteq f_i(x) \cap f_i(y)$ if $i = 2, 3, 4, 7, 8, 9$.

(g) f_i is idempotent if $i = 2, 5, 6, 7$.

(h) f_i is co-idempotent if $i = 3, 4$.

(i) $f_i \circ f_i = f_i$ if $i = 2, 4$.

Proof:
Clearly, properties (d) and (f) are simple set-theoretical results. Properties (a)–(c), and (e) follow from Theorem 4.1 and Proposition 1.1. (g) and (h) are consequences of (b) and Theorem 4.3. Thus, we only show (i). To this end it suffices to obtain $(f_4 \circ f_4)(x) \subseteq f_4(x)$ for any $x \subseteq U$. Indeed, $f_4 \circ f_4 = f_4$ by (h) then. Thanks to the duality of f_2, f_4 we can apply Proposition 1.1 and we finally obtain $f_2 \circ f_2 = f_2$. Now let $u \in (f_4 \circ f_4)(x)$. By definition, for each $w \in \tau(u)$, $I(w) \cap f_4(x) \neq \emptyset$. Consider any (A) $w \in \tau(u)$. Then there is (B) $v \in I(w)$ such that (C) $v \in f_4(x)$. By (B), $w \in \tau(v)$. Hence by definition and (C), $I(w) \cap x \neq \emptyset$. Thus by (A), $u \in f_4(x)$. □

The property (i) and Theorem 4.2 provide us with an argument that f_0 and f_4 are different in general, so are f_0^d and f_2.

In case (6) is satisfied (i.e., similarity is symmetric), some mappings become identical like in the case of f_0 and f_1. Then some other results can be enhanced as well.

Proposition 4.4. Assume (6). Then for any sets $x, y \subseteq U$, the following results can be obtained:

(a) $f_2 = f_0 \circ f_0^d = f_1 \circ f_1^d$.

(b) $f_3 = f_0 \circ f_0 = f_1 \circ f_1$.

(c) $f_4 = f_8 = f_0^d \circ f_0 = f_1^d \circ f_1$.

(d) $f_5 = f_6 = f_0^d \circ f_0^d = f_1^d \circ f_1^d$.

(e) $f_7 = f_0 \circ f_0^d \circ f_0^d = f_1 \circ f_1^d \circ f_1^d$.

(f) $\mathrm{id} \leq f_8$.

(g) $f_8 \circ f_8 = f_8$.

(h) $f_6^d = f_3$ and $f_8^d = f_2$.

(i) $f_9 = f_0 = f_1$.

(j) $f_9(x \cup y) = f_9(x) \cup f_9(y)$.

Proof:
We prove (i) only. Assume (6). In virtue of Theorem 4.3 (d) and Proposition 4.2, it suffices to show that $f_1(x) \subseteq f_9(x)$ for any $x \subseteq U$. To this end assume $u \in f_1(x)$. By definition $I(u) \cap x \neq \emptyset$. Hence there exists (A) $w \in x$ such that $w \in I(u)$. By assumption, (B) $u \in I(w)$. Consider any (C) $v \in I(w)$. By assumption $w \in I(v)$. Hence (D) $I(v) \cap x \neq \emptyset$ by (A). Thus there exists $w \in U$ such that $u \in I(w)$ and $\forall v \in I(w).I(v) \cap x \neq \emptyset$ by (A)–(D). By definition $u \in f_9(x)$. □

Summarizing, the considered mappings are related as follows.

$$f_5 = f_6 \leq f_7 \leq f_0^d = f_1^d \leq f_2 \leq \text{id} \leq f_4 = f_8 \leq f_9 = f_0 = f_1 \leq f_3. \tag{22}$$

As expected, stipulating that $I^{\rightarrow}(U)$ is a partition of U leads to the classical case. Moreover, the postulate (a6) is fulfilled.

Proposition 4.5. Assume $I^{\rightarrow}(U)$ is a partition of U. Then (a) $f_i = f_0$ for $i = 0, 1, 3, 4, 8, 9$; (b) $f_i = f_0^d = f_1^d$ for $i = 2, 5, 6, 7$; and (c) for any definable $x \subseteq U$, $i = 0, \ldots, 9$, and $j = 0, 1$, we have that $f_i(x) = x$ and $f_j^d(x) = x$.

Proof:
We only prove (a) and (b). The rest is left as an exercise. Assume that $I^{\rightarrow}(U)$ is a partition of U. By Theorem 4.2, $f_1 = f_3$. Also for each $x \subseteq U$, $(f_0 \circ f_0^d)(x) \subseteq f_0^d(x)$. To this end let $u \in (f_0 \circ f_0^d)(x)$. There is (A) $w \in f_0^d(x)$ such that (B) $u \in I(w)$. By (B) and assumption, $I(u) = I(w)$ and $\tau(u) = \tau(w)$. Hence by (A), $\tau(u) \subseteq x$, i.e., $u \in f_0^d(x)$. Thus $f_1^d = f_2$. Hence by (22) we have that: $f_5 = f_3^d = f_1^d$; $f_7 = f_1^d$; and $f_4 = f_2^d = f_1$. □

4.4. Discussion

Let us discuss the results. Assuming the postulates (a1)–(a6), we theoretically know the best candidate for the lower approximator (see (20)). The best candidates for upper approximators are $\leq$-minimal mappings of the form $f_0 \circ g$, where $g : \wp(U) \mapsto \wp(U)$ satisfies $f_0 \circ g \circ f_0 = f_0$ and $\text{id} \leq f_0 \circ g \leq f_1$. In practice it would be good to approximate sets by means of concrete low- and upp-mappings even if they do not fully satisfy the postulates.

From the mappings investigated so far, the mappings f_i where $i = 0, 1, 3, 4, 8, 9$ may be candidates for upp-mappings, while f_0^d, f_1^d, and f_i for $i = 2, 5, 6, 7$ may be candidates for low-mappings. The postulates (a1) and (a3) concern low-mappings, (a2) and (a4) describe some desired properties of upp-mappings, while the postulates (a5) and (a6) – the both kinds of mappings. In this paper we were mainly focused upon satisfiability of the postulates, treating other properties as secondary. The results are presented in Tabl. 1 (the general case) and Tabl. 2 (the case where (6) holds). By $+$ (resp., $-$) we denote that a condition is (is not) satisfied, while $\perp$ denotes that the result does not count. In the second column ("form") we give the description of mappings in terms of f_0, f_1 and their dual mappings. The values of the attribute "status" are u (upp-mapping) and l (low-mapping). The forth column contains the forms of the dual mappings. We omit the postulate (a6) since none of the considered mappings satisfy it unless $I^{\rightarrow}(U)$ is a partition of U.

In the general case where $I^{\rightarrow}(U)$ is merely a covering of U (Tabl. 1), the mapping f_7 is in our opinion the best candidate for a low-mapping since it satisfies the postulates (a1), (a3), and (a5). f_5 and f_6 have similar merits if compared to each other, yet they are apparently worse than f_7. The mappings f_1^d and f_2 are competitive to each other. Their usefulness depends on which postulate of (a3), (a5) is viewed as more important. Among the candidates for upp-mappings no mapping is the best. The mappings f_0 and f_4 seem to be competitive and the choice of one of them would depend on which of (a4), (a5) is more preferred. It is worth recalling that for $i = 2, 4$, $f_i \circ f_i = f_i$, while this property holds for f_0 (and hence for f_0^d) iff $I^{\rightarrow}(U)$ is a partition of U. On the other hand, unlike f_4, f_2, the mappings f_0 and f_1^d enjoy such properties as (j) and (m) in Theorem 4.1, respectively. The mapping f_9 is better than f_8. Unfortunately, the problem with the two mappings consists in that they are not increasing in general.

Table 1. Approximation mappings and satisfiability of the postulates in the general case

f	Form	Status	f^d	$a1$	$a2$	$a3$	$a4$	$a5$
f_0		u	f_0^d	$\perp$	$+$	$\perp$	$-$	$+$
f_1		u	f_1^d	$\perp$	$+$	$\perp$	$+$	$-$
f_0^d		l	f_0	$+$	$\perp$	$-$	$\perp$	$-$
f_1^d		l	f_1	$+$	$\perp$	$+$	$\perp$	$-$
f_2	$f_0 \circ f_1^d$	l	f_4	$+$	$\perp$	$-$	$\perp$	$+$
f_3	$f_0 \circ f_1$	u	f_5	$\perp$	$+$	$\perp$	$-$	$+$
f_4	$f_0^d \circ f_1$	u	f_2	$\perp$	$+$	$\perp$	$+$	$-$
f_5	$f_0^d \circ f_1^d$	l	f_3	$+$	$\perp$	$+$	$\perp$	$-$
f_6	$f_1^d \circ f_1^d$	l	$f_1 \circ f_1$	$+$	$\perp$	$+$	$\perp$	$-$
f_7	$f_0 \circ f_1^d \circ f_1^d$	l	$f_0^d \circ f_1 \circ f_1$	$+$	$\perp$	$+$	$\perp$	$+$
f_8	$f_1^d \circ f_1$	u	$f_1 \circ f_1^d$	$\perp$	$-$	$\perp$	$+$	$-$
f_9	$f_0 \circ f_1^d \circ f_1$	u	$f_0^d \circ f_1 \circ f_1^d$	$\perp$	$-$	$\perp$	$+$	$+$

Now consider the case the uncertainty mapping I satisfies (6) (see Tabl. 2). The mapping f_7 is still the best candidate for a low-mapping, while the mapping f_0 – which is actually identical with f_1 and f_9 – is the best candidate for an upp-mapping. Some authors advocate f_4 on the grounds that $f_4 \leq f_1$. It would be right unless the both mappings satisfy (a5). In our case however, f_1 does satisfy (a5) as opposite to f_4! This observation provides a new argument for preferring f_1 (or, equivalently, f_0).

5. Some related works

In this section we briefly recall some other generalizations of the classical notion of rough approximation.

Let us first mention Cattaneo's works [3, 4]. He considers an abstract approximation structure[8], based on some lattice with an ordering relation $\preceq$, for which he defines inner and outer approximation mappings, written i and o, respectively. The mapping i is decreasing, while o is increasing. For any approximable element x of the abstract structure, $i(x)$ is the greatest definable element such that $i(x) \preceq x$. On the other hand, $o(x)$ is the least definable element such that $x \preceq o(x)$. This part of Cattaneo's approach is thus focused upon the problem of definability of the results of approximation. Positive,

[8]Various approximation spaces in the sense of Cattaneo are simply realizations of this structure.

Table 2. Approximation mappings and satisfiability of the postulates if (6) holds

f	Form	Status	f^d	$a1$	$a2$	$a3$	$a4$	$a5$
$f_0 = f_1 = f_9$		u	f_0^d	$\perp$	$+$	$\perp$	$+$	$+$
$f_0^d = f_1^d$		l	f_0	$+$	$\perp$	$+$	$\perp$	$-$
f_2	$f_0 \circ f_0^d$	l	f_4	$+$	$\perp$	$-$	$\perp$	$+$
f_3	$f_0 \circ f_0$	u	f_5	$\perp$	$+$	$\perp$	$-$	$+$
$f_4 = f_8$	$f_0^d \circ f_0$	u	f_2	$\perp$	$+$	$\perp$	$+$	$-$
$f_5 = f_6$	$f_0^d \circ f_0^d$	l	f_3	$+$	$\perp$	$+$	$\perp$	$-$
f_7	$f_0 \circ f_0^d \circ f_0^d$	l	$f_0^d \circ f_0 \circ f_0$	$+$	$\perp$	$+$	$\perp$	$+$

negative, and border regions are implicitly taken into account when fuzzy rough approximation spaces are studied. We stop at this moment since fuzziness is out of scope of the present paper.

In Krawiec, Słowiński, and Vanderpooten's similarity-based rough set model [14, 28], the underlying key-concept is *ambiguity*, adapted to our framework as follows. Let $\mathcal{A} = (U, I, \kappa)$ be an approximation space.[9] An object $u \in U$ is *ambiguous* with respect to a set of objects $x \subseteq U$ if (1) $u \in x$ and there exists $v \in U - x$ which is similar to u or if (2) $u \in U - x$ and there exists $v \in x$ which is similar to u. Thus, there may be distinguished positive objects that do belong to x, negative objects that certainly do not belong to x, and ambiguous objects of two kinds. Having in mind such a motivation, f_0^d is taken as the low-mapping, while f_0 serves as the upp-mapping. In other words, $f_0^d(x)$ and $f_0(x)$ are considered as the lower and upper approximations of x, respectively.

Kryszkiewicz [15] as well as Stefanowski and Tsoukiàs [29, 30] study *incomplete* information systems (or tables), where some values of attributes are unknown. The notions of lower and upper rough approximations of sets proposed in their approaches are based on similarity (or tolerance) relations. The tolerance relation considered in [15] is symmetric, while the similarity relation proposed in [30] may or may not be symmetric. The lower and upper approximations of sets are defined as in [14]. Stefanowski and Tsoukiàs consider also a refined notion of a tolerance relation, viz., a *valued tolerance relation* [29, 30]. Then an approximation is viewed as a continuous valuation, and fuzziness comes into play also here.

Wong, Wang, and Yao generalize the rough set model (see, e.g., [35]) by considering a triple (U, V, γ), where U and V are finite sets and $\gamma : U \mapsto \wp(V)$ is a mapping, associating with every object $u \in U$ a set of elements of V that are in some sense *compatible* with u. It is assumed that for every $u \in U$, $\gamma(u) \neq \emptyset$ and for every $v \in V$, there is $u \in U$ such that $v \in \gamma(u)$. Every set $x \subseteq V$ may be described in terms of objects of U by means of lower and upper approximation mappings, $\underline{\text{app}}$ and $\overline{\text{app}}$, respectively, defined as follows:

$$\underline{\text{app}}(x) \stackrel{\text{def}}{=} \{u \in U \mid \gamma(u) \subseteq x\}.$$
$$\overline{\text{app}}(x) \stackrel{\text{def}}{=} \{u \in U \mid \gamma(u) \cap x \neq \emptyset\}. \quad (23)$$

Clearly, these ideas may be easily adapted to our case. It should be added that Wong, Wang, and Yao go further in the generalization of Pawlak's rough set model, viz., they consider two arbitrary Boolean

[9] In this case κ may be neglected.

algebras, $\mathcal{A}$ and $\mathcal{B}$, with universes A and B, respectively, and mappings $\underline{f}, \overline{f} : A \mapsto B$ satisfying some axioms. Pairs $(\underline{f}, \overline{f})$ are called *interval structures*.

In the last but not least model mentioned here, it is possible to approximate sets of objects to some degree. In our terms, Ziarko's *variable precision rough set model* [39, 40] may be described as follows. Consider an approximation space $\mathcal{A} = (U, I, \kappa)$, where U is finite, $(*)$ $I^{\rightarrow}(U)$ is a partition of U, and κ is defined by (8). Given lower and upper limit parameters $0 \leq s < t \leq 1$, for every set of objects $x \subseteq U$, we can define a t-positive region of x, written $\mathrm{POS}_t(x)$, an (s,t)-border region of x, written $\mathrm{BNR}_{s,t}(x)$, and an s-negative region of x, written $\mathrm{NEG}_s(x)$, as follows:

$$\begin{aligned} \mathrm{POS}_t(x) &\stackrel{\mathrm{def}}{=} \bigcup\{I(u) \mid u \in U \wedge \kappa(I(u), x) \geq t\}. \\ \mathrm{BNR}_{s,t}(x) &\stackrel{\mathrm{def}}{=} \bigcup\{I(u) \mid u \in U \wedge s < \kappa(I(u), x) < t\}. \\ \mathrm{NEG}_s(x) &\stackrel{\mathrm{def}}{=} \bigcup\{I(u) \mid u \in U \wedge \kappa(I(u), U - x) \geq 1 - s\}. \end{aligned} \tag{24}$$

In this case we get two parameterized families of low- and upp-mappings, $\{\mathrm{POS}_t\}$ and $\{g_s\}$, respectively, where for s, t, x as above, $g_s(x) = \mathrm{POS}_t(x) \cup \mathrm{BNR}_{s,t}(x)$. Clearly, $f_2 = \mathrm{POS}_1$ and $f_3 = g_0$. Now if we drop the assumption $(*)$, other parameterized families of low- and upp-mappings, generalizing the remaining mappings f_i ($i = 0, 1, 4, \ldots, 9$), can be obtained.

6. Summary

In this paper we considered the general case of rough approximation of sets of objects of some approximation space, where the uncertainty mapping generates a covering of the universe which may or may not be a partition of the universe. We formulated a few natural conditions (postulates) to be satisfied by approximation mappings. We have defined a mapping being theoretically the best lower approximator. Unfortunately, the best upper approximator has not been found yet. Looking for more practical results, we investigated several approximation mappings, some of them already known from the literature. They are intuitively simple to define in terms of an uncertainty mapping and the usual set-theoretical operations. We tried to decide which of the mappings might be the most appropriate for the purpose of general rough approximation. For we checked the satisfiability of the postulates and proved other properties. Also, a few related approaches were discussed. It would be interesting to adapt certain ideas (e.g., variable precision rough approximations or interval structures) to some of the approximation mappings studied in the present paper.

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